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EASY MATHEMATICS OF ALL KINDS

VOL. I. CHIEFLY ARITHMETIC

EASY MATHEMATICS

CHIEFLY ARITHMETIC

BEING A COLLECTION OF HINTS TO TEACHERS, PARENTS, SELF-TAUGHT STUDENTS, AND ADULTS

AND

CONTAINING A SUMMARY OR INDICATION OF MOST THINGS IN ELEMENTARY MATHEMATICS USEFUL TO BE KNOWN

BY

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"The parent inherits a primal tendency to revert to the fixed and rooted form, while the child is 'free-swimming': it is the natural explorer. And for ages we the parents through the teachers have been more and more successfully trying to train and educate our 'free-swimmers' into fixed and rooted prisoners; thus atrophising or mutilating their discovering and interpretative powers just as our own were injured at the same age."

LADY WELBY.

"There are several chapters in most arithmetic books that are wholly unnecessary . . . . but a writer of a school-book for elementary schools is not his own master; he must comply with the often unwise demand of teachers and examiners."

A. SONNENSCHEIN.

PREFACE.

THIS book is written without the least regard to any demand but those of children and of life and mental activity generally. In places where the author is mistaken he cannot plead that he has been hampered by artificial considerations. His object in writing it has been solely the earnest hope that the teaching of this subject may improve and may become lively and interesting. Dulness and bad teaching are synonymous terms. A few children are born mentally deficient, but a number are gradually made so by the efforts made to train their growing faculties. A subject may easily be over-taught, or taught too exclusively and too laboriously. Teaching which is not fresh and lively is harmful, and in this book it is intended that the instruction shall be interesting. Nevertheless a great deal is purposely left to the enterprise of the student and the living voice of the teacher, and the examples given for practice are insufficient. The author has usually found that examples and illustrations are likely to be most serviceable, and least dull, when invented from time to time in illustration of the principles which are then being expounded; but a supplementary collection of exercises for practice is necessary also, in order to consolidate the knowledge and establish the principles as an ingrained habit. Wearisome over-practice and iteration and needlessly long sums should be avoided; because long sums, other than mechanical money addition, seldom occur in practice, and especially because many kinds of future study, especially the great group of sciences called Natural Philosophy, will be found to afford plenty of real arithmetical practice; and even ordinary life affords some, if an open mind is kept. The cumbrous system of weights and measures still surviving in this country should not be made use of to furnish cheap arithmetical exercises of preposterous intricacy and uselessness. There is too much of real interest in the world for any such waste of time and energy.

The mathematical ignorance of the average educated person has always been complete and shameless, and recently I have become so impressed with the unedifying character of much of the arithmetical teaching to which ordinary children are liable to be exposed that I have ceased to wonder at the widespread ignorance, and have felt impelled to try and take some step towards supplying a remedy. I know that many teachers are earnestly aiming at improvement, but they are hampered by considerations of orthodoxy and by the requirements of external examinations. If asked to formulate a criticism I should say that the sums set are often too long and tedious, the methods too remote from those actually employed by mathematicians, the treatment altogether too abstract, didactic, and un-experimental, and the subject-matter needlessly dull and useless and wearisome.

Accordingly, in spite of much else that pressed to be done, a book on arithmetic forced itself to the front. It is not exactly a book for children, though I hope that elder children will take a lively interest in it, but perhaps it may be considered most conveniently as one continuous hint to teachers, given in the form of instruction to youth; and it is hoped that teachers will not disdain to use and profit by it, even though most of them feel that all the facts were quite well known to them before. It is not intended to instruct them in subject-matter, but to assist them in method of presentation; and in this a good deal of amplification is left to be done by the teacher. But it is of the first importance that the teacher's own ideas should be translucently clear, and that his or her feeling for the subject should be enthusiastic: there is no better recipe for effective teaching than these two ingredients.

For supplementary hints in connexion with the teaching of very small children, a subject which occupies the first four chapters, a couple of little books by Mrs. Boole recently published by the Clarendon Press may be mentioned; and as a convenient collection of suitable examples for practice I suggest a set by Mr. C. O. Tuckey published by Bell and Sons. For supplementary information on the higher parts of the work such a book of reference as Chrystal's *Algebra* is probably useful.

The author has to thank Mr. T. J. Garstang, of Bedales School, Petersfield, Hampshire, and also Mr. Alfred Lodge, of Charterhouse, late Professor of Pure Mathematics at Coopers Hill, for reading the proofs and detecting errors and making suggestions.

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PART II.

MISCELLANEOUS APPLICATIONS AND INTRODUCTIONS.

NOTES AND ERRATA.

The author is indebted to several readers for noticing one or more of the following numerical slips :

p. 69, line 2, *read* 5561 *instead of* 5651, twice.

line 4, *read* £9·36875 *instead of* £9·3875 ; the answer given is right.

p. 73, Example 18, *read* 52·8 *instead of* 51·12.

p. 126, The signs of inequality should obviously be replaced by signs of equality, if the statement of inequality is made in that form.

p. 173, last line but one, *read* 0·5 *instead of* 1·5.

p. 175, The cube root of ten is 2·1544347... ; and a few lines in the upper half of the page should be amended accordingly.

p. 218, middle, replace signs of multiplication by signs of addition.

p. 228, middle, *read* $\frac{a}{1-r}=\frac{\cdot 3}{1-\frac{1}{10}}=$ *instead of* $\frac{a}{r-1}$.

p. 245, last line but four, *read* $\sqrt{(ab)}$ *instead of* $\sqrt{(a^2b^2)}$.

p. 318, last line, *read* $(a-b)^{-m}$ *instead of* $(a+b)^{-m}$.

p. 322, last line, *for* $-1\cdot 6$ *read* $-1\cdot\dot{6}$.

p. 330, line 9 from bottom, "n" is redundant.

Lodge's Easy Mathematics.

CHAPTER I.

The very beginnings.

CONCERNING the early treatment of number for very small children the author is not competent to dogmatise, but he offers a few suggestions, the more willingly inasmuch as he is informed by teachers that a great deal of harm can be and often is done by bad teaching at the earliest stages, so that subsequently a good deal has to be unlearnt. The principle of evolution should be recollected in dealing with young children, and the mental attitude of the savage may often be thought of as elucidating both the strength and the weakness of their minds.

Counting is clearly the first thing to learn; it can be learnt in play and at meals, and it should be learned on separate objects, *not* on divided scales or any other *continuous* quantity. The objects to be counted should be such as involve some childish interest, such as fruit or sweets or counters or nuts or coins. Beans or pebbles will also do, but they should not be dull in appearance, unattractive as objects of property, and so not worth counting. The pips on ordinary playing cards will also serve, and they suggest a geometrical or regular arrangement as an easy way of grasping a number at a glance.

Counting should begin with quite small numbers and should not proceed beyond a dozen for some time, but there is no object in stopping or making any break at ten. Several important facts (the *facts* only, not their symbolic expression) can now be realised: such as that $3+4=7$, that $7-4=3$, that two threes are 6, and that three twos are the same, without any formal teaching beyond a judicious question or two. The lessons, if they can be called lessons, should go on at home before school age; but, whether this initial training is done at school or elsewhere, formal teaching at this stage should be eschewed, since it necessarily consists largely in coercing the children to arrive at some fixed notion which the teacher has preconceived in his mind—a matter usually of small importance. The children should form their own notions, and be led to make small discoveries and inventions, if they can, from the first. Mathematics is one of the finest materials for cheap and easy experimenting that exists. It is partly ignorance, and partly stupidity, and partly false tradition which has beclouded this fact, so that even influential persons occasionally speak of mathematics as "that study which knows nothing of observation, nothing of induction, nothing of experiment,"—a ghastly but prevalent error which has ruined more teaching than perhaps any other misconception of the kind.

As soon as small groups can be quickly counted, and simple addition and subtraction performed with a few readily grasped and interesting objects—and the more instinctively such operations can be done the better,—the time is getting ripe for the introduction of symbols—for that arbitrary and conventional but convenient symbolism whereby ⁙ is denoted by a crooked line, 5, and so on: a symbolism which the adult is only relieved from the necessity of elaborating and feeling difficult because of the extreme docility and acquisitiveness of childhood. It has already learned 26 symbols, it will patiently absorb nine or ten more, especially as they are soon found to be real conveniences; though if an adult wishes to realise the genuine difficulty of the process—always a most desirable thing to do—he should set to work to learn the Morse telegraphic alphabet, especially in the forms used for cable telegraphy.

I see no reason now why :: :·: should not be written $4+5=9$, or soon afterwards why : ·(:) might not be written $5-2=3$; but let no one suppose that these steps in nomenclature are easy. The nomenclature introduced is just as hard as that of trigonometry or the calculus, only adult persons are accustomed to the one and are often unacquainted with the other. A set of little blocks, or some simple cheap squared paper lends itself to statements like the following:

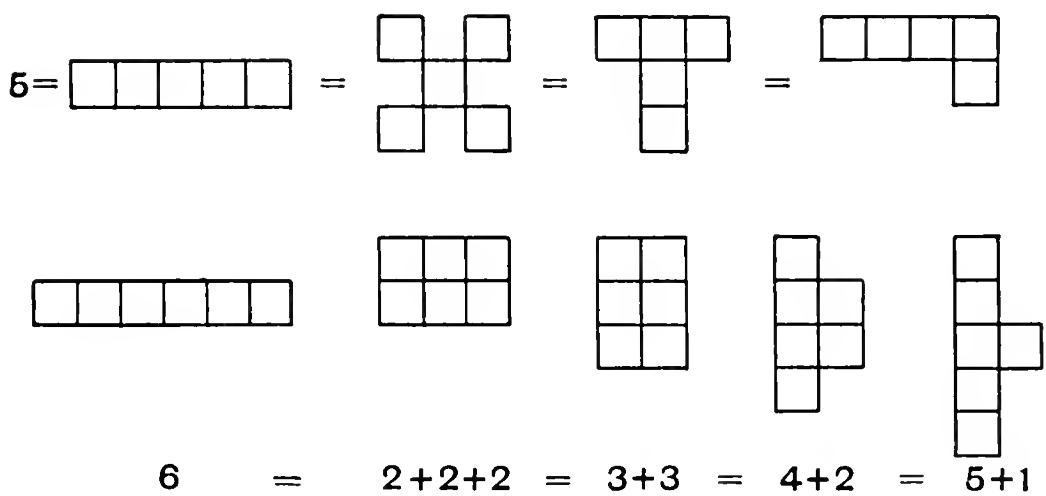

Fig. 1.

I see no reason for troubling about the *names* "addition" and "subtraction," nor yet for artificially withholding them. If they come naturally and helpfully, let them come. Nothing is gained by *artificial* repression at any stage. Premature forcing of names is worse than artificial withholding of them, but both are bad. If a gas, bubbling out of soda water and extinguishing a flame, is familiarly known as "carbonic acid," let it be called so: it is a help to have a label with which to associate observed properties, just as it is convenient to call a certain flower "daisy," or a certain star "Sirius." But to supply the label and withhold the object, to lecture about daisies or stars or numbers before they have been seen, is, let us politely say, unwise.

It seems to me that card games with counters may now be introduced, to enable the children to realise that their property may mount up beyond the smaller numbers that would be wholesome with sweets; and they can learn how to group their counters into packets of six, or even into dozens, and then they will have simply to count their packets and the odd ones over. A child with four packets of six and three over would have a real idea of his wealth, though "twenty-seven" might still be a meaningless expression.

Differently coloured counters are now serviceable to replace the packets, and thus the idea, but not the word, of different "denominations" will be imperceptibly arrived at: and it will be clinched by the at first unexpected discovery that even strangers will accept one white coin as equivalent to six much larger brown ones.

After this, some approach toward the admirable Arabic notation, whereby value is symbolised by place or position as well as by shape of digit, may be unobtrusively entered on. The idea of boxes or cases, or spaces of different value, in one of which odd counters or pennies are to be stored, another one in which packets, or silver coins, are to be kept; and ultimately, but not too soon, a third one which is to be occupied by packets of packets, or gold coins; if ever such wealth were attained.

While there is every advantage in thus emphasising attention to the value or place of the digit, and so to a system of numeration, there are many reasons against concentrating attention on the particular number "ten" prematurely: it is not a specially natural number, for one thing; for another thing it is so large that ten packets of ten are unlikely to occur, whereas four packets of four, or six of six are quite possible. Another reason is that it is undesirable to suggest, what habit will subsequently only too erroneously enforce, that there is something special and divine about the number ten, so that the arrangement of digits 12 cannot help meaning a dozen. This false idea, due merely to habit, will not occur to a child, nor will he know intuitively that twelve pence make a shilling, or twenty shillings a sovereign; indeed, strange to say, he is usually somewhat callous as to the importance of this pivot of human existence; and, though he soon gets to like coins, he attends chiefly to their number without much regard to their denomination, unless some are specially new and bright.

Having got so far, the conventional symbolism, in which practice has been quietly going on in the background during the few more formal school quarter-hours, may be extended, and the digit-symbols written in spaces drawn to represent the boxes, or on paper ruled into quarter-inch squares, which is cheaply and plentifully accessible, so that a 4 put in one box shall signify 4 counters, while a 4 put in another box shall signify 4 packets of say ten counters each, so that at the end of a game | 0 | 3 | shall mean that the loser has no packets and only three counters altogether, while another child may have | 3 | 0 |; that is, three complete packets and none over. A third may have two packets and five over; that is to say | 2 | 5 |, and another, the winner at the game, may possess | 1 | 5 | 2 |, or in words, 1 packet of packets, 5 simple packets, and 2 odd ones.

The packets may be represented by otherwise coloured counters, or the well-known Tillich bricks or other Kindergarten devices can be employed for convenience; the important thing is not prematurely (*i.e.* not until the underlying reality has been essentially grasped) to proceed to the only partially expressive symbolism 25 or 152, which to us by mere habit looks so living and significant. Let the elementary teacher reflect that to a mathematician the symbol $\int_0^\infty e^{-x^2}dx$ looks equally living and significant, and be not hasty with the children.

At the same time there is no need for artificial delay. A child brought along the right lines will jump forward without difficulty, will recognise the **places** without the boxes, will get accustomed to the savage's mode of reckoning by tens without being encouraged to go through the savage process of counting on his fingers, and before long will be able to interpret such a complicated symbolism as 50327, or £175. 16s. 11d. The last, indeed, is properly spoken of as "compound" instead of simple, for in it "scales of notation" are badly mixed up. The reckoning proceeds by tens, by dozens, and by scores, sometimes one and sometimes another, occasionally by quarters also.

The poor child who finds himself able to master this and the operations which arise out of it, need not be deterred by any legitimate obstacles in mathematics until he comes to its really higher walks, beyond simple differential equations: a step which he will not be called upon to take at all unless he is born to be a mathematician, in which case difficulties of any ordinary kind will barely be felt.

The operations of addition and subtraction may now be extended. $7 + 5$ may be done into a packet of one dozen, or into a packet of ten and two over, and denoted by 1/- or 12 according to which plan of grouping is adopted.

So also $8 + 7$ may be called either 1/3 or 15, the former being the custom if they are pennies, the latter if they are nuts.

It is necessary to apologise to children for this needless complication; but they inherit some things that are good, to make up for several things that are stupid, and therewith they will have to be content:—

8 + 7 + 9, if shillings, will be grouped differently again, and be denoted by £1. 4s.; if pennies, they will be denoted thus, 2/-; if ounces, they will be written 1 lb. 8 oz.; if feet, they will be called 8 yards; if farthings, they will be written 6d.; if oranges, they will be called 2 dozen; but if boys, they will be written 24.

I do not recommend anyone to confuse the minds of children by pointing out these anomalies, or by quoting a sample of them simultaneously as above. Children will not detect their true character, but will docilely receive them as if all this rubbish were part of the laws of nature. This may account for their disinclination later on to make acquaintance with any more of those laws than they can help, but at this stage they are docile and assimilative enough: they can at this stage be taken advantage of with impunity. But I should very much like to confuse the minds of some teachers, and of some school inspectors—especially some varieties of school inspector and university examiner—and get them into a more apologetic and humble mood at having to insist on filling the mind of a child with any more of these artificial insular conventions than is absolutely necessary in the present stage of British political and commercial wisdom.

It is undesirable to hasten forward to numbers involving 3 digits too quickly; they can be mentioned and illustrated when convenient, but real **work** should for some time be limited to 2 figure numbers, because in these the real principles can be recognised and grown accustomed to in the simplest way.

The early operations in which practice can be given are such as the following: Suppose counters are employed and that little cases have been made which just hold six or ten or any convenient number, suppose ten:

Then 13 will stand for one packet of ten and three counters over;

17 added to it will amount to two packets and ten counters over; which the child, if encouraged by the sight of an unused case available, may wish to make up into 3 whole packets, and so recognise the propriety of denoting the number by

30

Similarly 15 + 17 will make up into three packets and 2 over, which may be shown thus:

	tens	ones
	1	5
	1	7
make	2	12
which equals	3	2

while 25 + 37 will equal five packets and twelve over, or six packets and 2 over; 29 + 37 = 66, but it is equally permissible to keep it as 5 packets and 16 counters over, if it should happen to be convenient—as it sometimes is.

To take 4 from 17 is easy,

but to take 9 from 17 will involve emptying a case; and only 8 counters will be left.

To take 13 from 25 can be done by removing 1 case and 2 counters;

to take 15 from 25 is also easy;

but to take 16 from 25 involves the breaking up of a packet.

After a time these operations can be followed when nothing concrete is present; but abstractions are not natural to children, and before calling upon them to follow a difficult conventional subtraction sum like

$$\begin{array}{r} 82 \\ 37 \\ \hline 45 \end{array}$$

the operation of breaking up packets should be introduced into the symbolism which is employed to faintly shadow the concrete reality.

It is perfectly right to speak of 3 packets and 13 loose counters, although they may be more compactly grouped as 4 packets and 3 counters. So if we have to subtract say 7 from 43 we shall first break up one of the four packets, so as to turn 43 into 3 packets and 13, and then subtract the 7 without difficulty, leaving what is abbreviated into 36.

Hence before doing the above conventional little sum, 8 packets and 2 should be expressed as 7 packets and 12, or |7|12|. From this 3 packets and 7 have to be removed, leaving obviously 4 packets and 5. Wherefore $82 - 37 = 45$ without any argument.

The abbreviated form of the above breaking-up operation, called borrowing, will now gradually almost suggest itself, if many sums of the kind are given to be done. But the best and easiest method of subtraction is the complementary method, and if this is taught from the first, the complexity of borrowing becomes unnecessary.

The adult cannot too clearly realise that many of the operations to which he has grown accustomed are labour-saving shorthand devices with the vitality and principle abbreviated out of them; quite rightly so for practical purposes but not for educational purposes. The race invented them at first in more elaborate shape, and gradually abbreviated them into their present-day form. The child will likewise get accustomed to this form in due time, but he should not be over-hurried into it.

After adding two numbers for some time we may proceed to add more than two,

and find that $7+9+6 = 22$, etc.;
also that $7+7+7 = 21$,

and it is natural to speak of this as three sevens.

So also the fact that $5+5+5+5 = 20$ will naturally be quoted as four fives make twenty; and thus the essential idea of multiplication will arrive, as a shorthand and memorised summary of the addition of a number of similar things, without any use of the name multiplication or any feeling of a new departure. To find the value of three seventeens, that is, to group them into tens and ones, is a problem for an afternoon, and if it be done with counters in the first instance, and ultimately with symbols, the meaning of the operations having been realised beforehand with the counters, so much the better.

The operation of adding or multiplying means grouping the whole number into tens and ones, or into hundreds, tens, ones, etc., instead of in the given groups.

A child must not be expected to be able to formulate his conception of the operations, or to express them accurately in words, at this stage. It is a capital exercise later, but it is enough at first for him to realise the meaning of what he is doing in the back of his mind. From time to time he can be encouraged to interpret processes into words, but they must have become familiar first. To be able to apply a rule, from a precise statement in words of what has to be done, is an adult accomplishment, often not reached by adults. To dissect out and state a rule in words, from a knowledge of what the operation really is, is perhaps easier, and is a desirable gift, but it is a training in the use of language rather than in the subject matter of the craft. It is most appropriate and valuable practice for children at the proper stage, a stage reached much earlier with some children than with others. Children who reach the word-expression stage late are usually called "stupid." If this adjective implies a stigma it is usually undeserved. There is a performance appropriate to each stage of development, and opprobrious epithets are generally employed by those who seek to force things several stages too soon. A highly trained and clever dog would soon prove himself "stupid" if tested by a formula, or by words even of only 3 letters. An adult who can hum or whistle an air may be told that he ought to be able to sit down and write it in the recognised musical notation. Similarly he *ought* to be able to read off a piece of music handed to him. He might resent being called stupid if he found it difficult to do these, to some, so simple things.

"Badness" of many kinds may exist in spoiled children (and there are several ways of spoiling them), but badness in unspoiled children is rare, and stupidity is almost non-existent unless they are physiologically out of order and therefore mentally deficient. Stupidity is however a product easily cultivated by improper feeding, especially improper mental feeding. The "badness" of children is largely the effort which nature makes at self-preservation; for inattention and laziness are the weapons whereby an attack of mental indigestion can be warded off.

The only fault with very young children is that they are too good, and therefore too easily damaged. Later on, a spirit of rebellion acts as a preservative, but it would be better to dispense both with the rebellious spirit and with the causes which necessitate it.

Returning from this digression, which is either false or else of very extensive application, to our immediate subject, viz., the introduction of the fundamental operations to be performed on number,—and remember that what are called the first four simple rules are tremendously fundamental and important, more important than anything which follows, until involution, evolution, and logarithms are arrived at,—we must exercise children in Multiplication and teach them something of the multiplication table, at first experimentally, but afterwards by straightforward memory work, for it is one of the things with which the memory may be rightly loaded. We can next recognise that Division too can be unceremoniously introduced by trying to split up numbers into equal parts. The endeavour to share sweets or fruit or cards or counters is an obvious beginning. Then, since children are docile, they can be asked to split up 2 packets and 7 into three equal groups, or they can be asked to split up 2 packets and 4 into eight equal groups, and so on; for no reason assigned. But it must be recognised that the operation of division in general is rather hard, and involves a good deal of tentative procedure or guess work. In other words it involves the rudiments of experiment and verification. Gradually, when the multiplication-table is fairly known over some little range, children can be encouraged to apply theory before practice and actually to think out the result before trying it; but this is a lesson in deductive reasoning, and represents the nascent beginnings of a loftier mode of procedure than ordinary adults are accustomed to apply to their affairs. When asked to split 28 into four equal heaps, it is an application of pure theory to remember that 4 sevens are 28 and then to count out seven counters into each heap at once. The empirical mode would be a method of **dealing** out singly into four groups and then counting the result. It is easily done with ordinary playing cards, but its value as training is much enhanced if theory is applied first.

If for instance 30 cards were given, to be dealt to four players, the residue that will not go round to be put in the middle or pool, a decided effort is required for a child to perceive that there will be two for the pool and seven for each player: but if he could have time allowed him so to think it out, and then to make the experiment, he would be conscious that his powers were developing, and he would in reality be introduced to the first beginnings of a mode of comprehending nature such as is in the higher stages reserved for men of science,—using the term science in its most comprehensive signification.

It is very often a mistake for teachers to suppose that some things are easy and other things are hard; it all depends on the way they are presented and on the stage at which they are introduced. To ascend to the first floor of a house is difficult if no staircase is provided, but with a proper staircase it only needs a little patience to ascend to the roof. The same sort of steps are met with all the way, only there are more of them. To people who live habitually on the third floor it is indeed sometimes easier to go on to the roof than to descend into the basement. Educators should see that they do not forcibly drive children in shoals up an unfinished or ill-made stairway, which only the athletic ones can climb. It is extremely difficult in familiar subjects not to go too fast. The effort sometimes results in a process of going *too* slowly, which is wearisome and depressing and the worse fault of the two.

Extension or Application of the idea of number to measuring continuous quantity.

So far we have been employing number to count discrete objects, and to perform simple operations of addition, and the like, among them. It is now appropriate to introduce the idea of multiples of a unit, so that one thing can be twice as long or twice as heavy as another, without being in another sense "two" at all. The lines on ruled paper enable one easily to draw across them a line twice or three times or six times as long as another. So also letter-scales can be used to show that a penny is twice as heavy as a half-penny, that a half-crown weighs how many sixpences, and the like.

Given a foot-rule they can measure the size of furniture, or of books. Given a few ounce weights they can make very rough estimates of the weights of things that have or might have to go by post.

It is desirable not to **dwell** on these things at this stage, but simply to accustom a child to recognise a rod 6 inches long, and such like, and to see instinctively and without formula or expression that *number may be applied to continuous magnitude by the device of a unit of measurement.* Adults may realise that there is a real step here, by remembering that if they were set to express the strength of an electric current, or the electric pressure on a main, or the strength of a magnet, numerically, they would be nonplussed, unless they knew something about the units which within a generation or two have been introduced for the purpose,—the ampere, the volt, and the line of force; so that nowadays the British workman is able to speak familiarly of an electric current of so many amperes—(sometimes pronounced "hampers"). There is nothing really *numerical* about the length of a table or the height of a door or the weight of a sack or the brightness of a lamp or the warmth of a room or the length of a day; and its numerical expression will depend entirely upon what conventional unit is employed, and may vary in different countries accordingly. Do not assume therefore that a child is stupid to whom an application of arithmetic to weighing and measuring is not obvious.

Introduction of the idea of fractions

In the same way the idea of fractions can naturally occur; a halfpenny and a half ounce and a half inch being fairly easy examples: but not the easiest. There can be no doubt that just as numbering ought not to begin with continuous quantity but with discrete objects, so fractions should be first displayed as actually cut and broken things.

The proper fractions to begin with are halves and quarters and eighths; and apples do admirably for that. Oranges suggest further modes of subdivision, except that the removal of the peel may constitute an unexpressed but felt complication.

Folding of a ribbon or paper easily leads to thirds and any other fractions wanted. Any child can be sent to cut off a quarter of a yard, or a yard and a half, or even a foot and three quarters, of tape. But again do not be surprised if this last mode of specification is found occasionally puzzling: it is of the nature of a problem, and requires time. The form of difficulty which may properly occur to some children is "a half of what" or "three quarters of what": and if they bring the foot and the 3 quarters all separate, *i.e.* if they cut the tape into four pieces altogether, that is very well for a beginning. They should not be supervised or fidgeted during the solution of a problem. They cannot think if they are. These expressions, 6 miles and a half, etc., have a conventional ring, to which we have grown thoroughly accustomed, but they are shorthand terms not really fully expressive: it might possibly ambiguously suggest 9 miles.*

The measure of time in half and quarter hours may also be

*Cf. George Meredith's "Rhoda Fleming," Chap. 3.

appealed to as illustrative of fractions; but in this form they are somewhat abstract. The divisions on a foot rule or metre scale are easier, and for further progress are indeed the easiest illustration to be borne in mind. Afterwards, the halfpenny, the halfcrown, the halfsovereign, etc., and the other fractions of money may be brought in, whenever they appear to be natural.

Practical hints for teaching the simple rules.

Simultaneously with all this introduction of fresh conceptions, mechanical practice in operations with symbolised numbers can be proceeded with :—

Addition.

About addition there is little to be said : the idea of packets must have made everything concerning the carrying-figure easy.

The principle being understood, it is now only a question of practice in attaining quick and sure execution, as quick and sure as it is worth while to aim at at this stage.

Addition of money is a useful accomplishment, and since the packets into which it is to be made up are varied, it affords good practice, involving a certain amount of constant thought and care. It is wrong to try to force a child to acquire the facility of a bank clerk in adding up long columns : that will come in due time and is quite a useful faculty : it is clearly a thing to acquire in commercial schools, but not while still young and receptive.

It is well to begin thus :

£	s.	d.
6 .	15 .	3
5 .	4 .	9
12 .	— .	—

where the packets to be carried forward are complete. Then change the 3 into a 4 or 5 and get 1 or 2 pence over; then change the 15 into 16 or 17 and get some shillings over, and so on, gradually. *Always begin with what illustrates the procedure in the simplest form and gradually complicate it.*

There is one remark about addition worth making. In adding say $43+8$, some beginners are told to bethink themselves that $3+8 = 11$, and so arrive at the digit 1 of the result; while others are told to think of the sum as $43+7+1$, stepping on to the intermediate stage of the complete packet *en route* to 51;

$$e.g.\ 77+9 = 77+3+6=80+6=86.$$

Perhaps it is permissible to introduce this aid as a temporary measure, but ultimately addition ought to proceed by instinct and without thought. It is a mechanical process, and a bank clerk who stopped to think, while adding, would be liable to make a mistake.

Subtraction.

There appears to be no doubt now but that the "shop method" of subtraction is the handiest and quickest: it may as well, therefore, be acquired almost from the first.

37	Three and **four** make seven.
13	One and **two** make 3.
24	Put down the figures in black type.

Verify by adding 13 to 24. Take another example:

174	
98	Eight and **six** make fourteen.
76	Nine and one and **seven** make seventeen.

I do not think that children need find this method hard or unnatural, but practice will be needed before going on to money sums, such as:

£	s.	d.
17 .	6 .	11
11 .	8 .	4
5 .	18 .	7

Four and **seven** make 11.
Eight and **eighteen** make 26.
12 and **5** make 17.

Verify by addition of the two lower lines. Get the children never to pass and hand in a result as finished unless they have taken pains to assure themselves that it is **right.** This does not mean that they are not to hand in a confessedly unfinished sum if they find they cannot do it without help.

Multiplication.

At good Kindergarten schools, a step beyond the first in multiplication is often introduced by some such questions as this:

How many stamps will three children have if each has 14?

They first add 14 three times, and they are allowed to do that till they find it quicker to use the phrase "three times," which, if they know the multiplication table, they can hardly help doing in the process of adding; and so they get to be able to give the answer "3 times 14" instantly, without necessarily having had time to realise what the operation would result in when executed. This kind of intermediate answer is to be encouraged.

In entering upon multiplication, employ a single digit as one factor, and do it first as an addition sum, *e.g.*:

$$\begin{array}{r} 142 \\ 142 \\ 142 \\ 142 \\ \underline{142} \\ 710 \end{array}$$

then proceed

142	173	125	£. s. d.
			12 . 7 . 6
5	5	8	4
710	865	1000	49 . 10 . —

doing this latter also by addition first:

£.	s.	d.
12 .	7 .	6
12 .	7 .	6
12 .	7 .	6
12 .	7 .	6
49 .	10 .	0

but it is well to lead up to the last type of sum by simple cases first, *e.g.* 4 × 2/6 = 10/-; 4 × 5/- = £1; 4 × 7/6 = 30/- = £1 10/-; 4 × 3d. = 1/-; 4 × 1/3 = 5/-; 4 × 10/- = £2; 4 × 11/3 = £2. 5s.

Do not hurry. If the child can be allowed time to see a connexion between the three last statements, or the like, so much the better. The value of these trifles is when they are discovered; there is hardly any virtue in them if they are pointed out, and none at all if they are laboriously emphasised. If they are not glimpsed let them pass. We all of us doubtless miss discoveries, most days, for lack of attention and insight.

Next comes multiplication with two digits: first by numbers like 10, 20, 70, etc.

Multiplying by ten means making every unit into a packet, every packet into a set of packets, and so on.

Wherefore

| 0 | 1 | 3 | 4 | when multiplied by ten becomes | 1 | 3 | 4 | 0 |,

the 1 being shifted into the empty compartment, and every other digit likewise moved; the unit box, or box for single counters, being left empty.

If we multiply by 20, the shift takes place similarly, and also every digit is doubled, yielding 2680.

So now start multiplying a number like 53 by 20, getting 1060.

Then a number with a carrying figure from the units place, like

$$47 \times 20 = 940\,;$$

then one involving two carryings, like

$$57 \times 20 = 1140,$$

and so on.

Next take multiplication by a number like 23. Let it be realised once more that 23 is short for $20 + 3$, so that it may be felt to be natural to multiply by 20 and by 3 successively and add the results, which is what we do. At first let it be worked in this way; for instance, to find

	824×23		
	$= 20 \times 824$	or	16480
	and 3×824	or	2472
added together make			18952

but gradually get it abbreviated into the usual form

```
  824
   23
 ----
 1648
  2472
 -----
 18952
```

without necessarily putting in the cipher after the digit 8.

There appears to be no doubt now that it is best in multiplication to begin with the most important figure, so that sums look thus:

```
  173        173        768
   56        156        107
 ----       ----       ----
 865        173        768
 1038        865        5376
 ----        1038      -----
 9688       -----      82176
            26988
```

a trivial matter to all appearance, but helpful in later stages, and therefore better practised from the first.

[In my opinion it is thoroughly unwise to reverse the digits of any factor before multiplying with them, though some teachers of immense experience think otherwise.]

Multiplication of money, at least of English money, is more difficult of course, because, in the specification of money, scales of notation are so mingled; thus, depicting the compartments and labelling them when necessary:

£			s.		d.	
4	3	5	1	7	1	1

at the double line the scale is changed from ten to a dozen, and at the treble line it is changed again from ten to a score.

So if we have to double this sum, even doubling it is complicated, and results in

£871 . 15 . 10

Let no one suppose that this is an easy process, for a child or anyone.

It could in this case be performed more easily by simple addition:

£	s.	d.
435 .	17 .	11
435 .	17 .	11
871 .	15 .	10

but that is hardly applicable to larger factors. Not only is doubling hard, but multiplying even by 10 is hard too. Take the amount £5. 17s. 11d. and multiply it by 10; it becomes the totally different-looking amount

£58. 19s. 2d.

Multiplying by 12 will of course turn all the pence into shillings, and multiplying by 20 will turn shillings into pounds, but multiplying by any other factor is hard, and is probably best deferred for the present.

If multiplication of money by a number like 23 is wanted, not only must the 23 be divided into two parts $20+3$, and the multiplication done separately as usual, but it is generally needful to resolve the 20 into two parts also, say $10+10$, and then add the three results together.

If however multiplication by 24 were desired, it would be possible to split it into two factors 8×3, and to multiply first by one and then the result by the other, without any addition of results; but there is great danger of confusion here, and there are plenty of what are considered and are really "higher" parts of arithmetic which are much easier than this. Low class or unskilled labour is not necessarily easy: it may in some cases be terribly laborious, like unloading a ship. Another way of multiplying by 20 is to split up 20 into the two factors 2×10 or 4×5 and employ them successively. In that case the result of multiplying by 23 is ultimately obtained by multiplying the original sum by 2, the result by 10, the original sum by 3, and then adding the last two results.

The fact is that with money specified in the customary English way, the only operations that can comfortably be performed on it are addition and subtraction, and these are the only really frequent operations in practice.

To apply multiplication and division it is best to express the money differently, in fact to decimalise it before commencing operations. This will be explained later (Chap. VII.), though of course to most teachers it is a process already well known. It ingeniously evades the difficulties caused by our currency, and converts its treatment into almost a worthy intellectual exercise.

Division.

First take simple sums to introduce the notation, such as

$$\frac{21}{7} = 3, \text{ or } 21 \div 7 = 3.$$

Let it be realised also that $\frac{21}{3} = 7$, and that $3 \times 7 = 21$.

There are a multitude of interesting things to be learnt before long about factors, and criteria for division, etc., but not yet; let the child learn how to perform the process on numbers of which he knows no factors. But at first do not trouble him with remainders: let him at first be given simple sums that divide out completely.

Thus we can tackle such sums as

$$7 \underline{|491036}, \quad \text{which should be also written } \frac{491036}{7} = 70148.$$
$$\;\;70148$$

The treatment of remainders is for subsequent consideration.

It is well to give the complementary sum 7×70148, especially since the teacher will thus have but little trouble in checking results—at least until the child finds out the dodge—a discovery which is to be encouraged like all other discoveries.

At good Kindergarten schools, a step beyond the first in division is often introduced by some such plan as the following:

To prove that $96 \div 4 = 24$.

Take nine bundles and six sticks over, deal out into four places, two bundles in each place; and then deal sixteen sticks, four into each place, giving the result 24. And so on with other numbers.

As soon as short division is thoroughly understood, long division may introduce itself as an assistance when more difficult divisors are involved; for instance $988 \div 19$. This being difficult to do by short division, where the multiplication and subtraction have to be done in one's head, it is permitted to write the operations down, at first both of them, thus:

```
19 ) 988 ( 5
     95
     --
      3
```

Afterwards, perhaps, only the result of them, 3, which in short division would likewise not appear, nothing but the quotient being written in short division. Long division is therefore not harder than short division, but easier: it is the identical process, only written out more fully, so as to be applicable to harder sums. It is the largeness of the figures dealt with that makes it hard.

For long division it appears to be felt that by aid of the shop system of subtraction there is no undue strain on the brain by the use of the abbreviated method.

I would have it understood however that long division sums are among the moderately hard things of life, and that mathematicians seldom trouble themselves to do them. They can be deferred until many other things have been done and some familiarity with figures acquired. It is a gymnastic exercise to perform even so simple a long division sum as the following, and if attempted too early will involve strain.

```
72 ) 5286456 ( 73423
      246
       304
        165
         216
```

This is the process:

Sevens in 52? guess 7 times and write 7 as the first digit in the quotient, then $7 \times 2 = 14$, to which add **4** to make 18. Seven sevens $= 49$, say 50, to which add **2** to make 52; record only the figures here printed in black type; bring down the rest of the dividend 6456 or as much of it as is wanted; only 6 is wanted so far, and we guess 3 for the next digit in the quotient. Three times 2 and **0** make 6, three times 7 and **3** make 24. Bring down more of the dividend, say 456, or at least 4, and guess 4 for the next digit.

$4 \times 2 = 8$ and **six** are 14.

$4 \times 7 = 28$, say 29, and **1** are 30.

Bring down the 5, and guess 2 for the next digit of the quotient; twice $2 = 4$ and $\mathbf{1} = 5$, etc., and then finally bring down 6, and it goes 3 times exactly.

If the sum is neatly done the corresponding places are vertically under each other, a detail of appearance emphasised by the presence of a decimal point.

Let the result be written

$$\frac{5286456}{72} = 73423.$$

Do not forget to set also the complementary sum

$$72 \times 73423.$$

It will be well also to set the exercise whose result is $\frac{5286456}{73423} = 72$, as a separate sum.

If the connexion is automatically noticed, it is well; it will prepare the mind for the later-on extremely important and constantly occurring connected relations,

$$\text{if } \frac{a}{b} = c, \text{ then } \frac{a}{c} = b, \text{ and } bc = a,$$

but refrain from using this abstract language at present. Watch for the time when it can without strain be naturally introduced. It is a great help when that step is reached, and it represents a vital stage of real mental progress. The mind should be soaked with particular instances however before generalisations can be usefully and permanently grasped.

Division of money is of course difficult, even when the divisor is a small number, because of our complex system of notation, unless the money is first expressed in decimal form.

To divide by 23 moreover it is not **correct** to divide by 20 and then by 3 and add the results, as it was with multiplication. A long-division sum is necessary, and that is no joke with money as usually specified. Division by 24 can indeed be done in two stages, by help of its factors 3 and 8 consecutively applied, but that only masks the essential difficulty by a device applicable only to special cases.

My object in introducing these remarks about complex money-sums here (and the same thing applies to weights and measures sums) is to urge that they really belong to a later stage, and to beg teachers to defer them beyond the early years at which they are too often introduced. For their premature employment has often resulted in giving children an effectual and lifelong disgust with what they have docilely conceived to be arithmetic; whereas much of what they had to do was really a mechanical and overstraining grind, having as much relation to mathematics as carrying heavy hods of bricks all day up a ladder has to architecture.

Origin of the symbols.

It is amusing to speculate on the probable origin of the symbols for the digits. It appears likely that if a single horizontal stroke meant 1, a double horizontal stroke hastily drawn would give Z or something like a 2.

It is less easy to make a sort of 3 out of three such strokes, but it is possible.

The symbol for four would seem to be representative of a four-sided figure or badly drawn square, , and the figure 8 was probably originally a pair of such squares .

But at this stage it appears likely that some skilled person took pains to design digit symbols of distinctive form by combination of a stroke and a semi-circle, making a set like this:

 etc.,

and that the notion of the value of "**place**" was a development from the further stages of this mode of representation.*

So also it is believed that the Roman symbol X for ten was the result of counting by strokes and crossing off every tenth stroke, thus:

/ / / / / / / / / X / / /

a practice not unknown among workmen to this day.

Two such crosses would naturally mean 20, etc., while half a cross or V could conveniently be used to denote 5.

It has been suggested that the rounded M for 1000, M, sometimes inscribed CIƆ, if halved, would give the D for 500, and that a square C for 100, if halved, would furnish an L for 50; but this may be fanciful. The symbol CCIƆƆ was used, it is said, for 100,000, and CCCIƆƆƆ for a million.

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* The above however is not history. The real history of the symbols is complex, and stages of it are given in Dr. Isaac Taylor's learned work on the Alphabet, especially Vol. II. pp. 263 *et seq.*

It appears that our digit symbols originated in India, and that several of them, especially 7, represent a corruption of the initial letters of the *words* previously employed to denote the numbers.

"They were introduced by the Arabs into Spain, from whence during the 12th and 13th centuries they spread over Europe, not, however, without considerable opposition. The bankers of Florence, for example, were forbidden, in 1299, to use them in their transactions, and the Statutes of the University of Padua ordain that the stationer should keep a list of the books for sale with the prices marked 'not by *ciphers* but in plain letters'. . . . Their use was at first confined to mathematical works, they were then employed for the paging of books, and it was not till the 15th century that their use became general."

CHAPTER II.

Further considerations concerning the Arabic system of notation, and extension of it to express fractions.

HAVING become acquainted with the fundamental plan of the system of notation in use, and the mode of expressing any whole number of things by a combination of ten digits arranged in places of different value, not all places necessarily occupied—that is, by means of nine significant digits and a cipher to express emptiness in whatever place emptiness may occur,—it is permissible to elaborate it further, with a little repetition occasionally.

At the beginning of each chapter there is liable to be a little repetition of something that has already been explained, but in a slightly different form. This amount of repetition is purposely introduced and is useful: it is intended to link the new knowledge on with the old. A new subject should not be introduced as if it belonged to a perfectly distinct region of thought; its connexion with what is known should be indicated, and sufficient of the old should be reproduced to make the connexion secure. Repetition of a judicious kind is by no means a thing to be avoided, though it is easy to overdo it; and in every way the best kind of repetition is that which repeats the old idea in a different form of words, or which looks at something already known from a new aspect.

The beginnings of each new chapter should be easy, and the steps to higher flights should be regular and moderate, like a staircase.

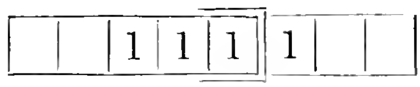
Now we know that the symbol 304 means usually that there are
3 packets of a hundred things each
no packets of tens and
4 single things,
but the "ten-system," though customary, is not an essential part of this plan of notation.

40 and 4/- are both constructed essentially on this plan, both are understood to signify 4 packets and no odd units, though the number in the packets is not the same in the two cases.

£4 . — . — signifies again 4 of another variety of packet.

Three dozen and six pennies may be written either 3/6 or 42 pence. It would have been far more convenient if the human race had agreed to reckon everything in dozens, and so to express this number by the digits 3 6 instead of by the digits 4 2; but as they have in early semi-savage times arranged otherwise, we must now make the best of it. The general idea is the same, only that whereas in ordinary life things are commonly and conveniently reckoned by dozens, it is customary in arithmetic to reckon by packets of ten, the symbols being called digits because they used to be reckoned by actual fingers: which by some simple persons are so employed still. Thus whereas 7/6 is understood to mean seven dozen and six pence, it is customary to mean by 76, seven packets of ten and six units over; that is to say, if the units were pennies, the same as 6/4. So also, instead of grouping dozens into a gross, as in ordinary life, in arithmetic we group tens into a large packet of ten tens, which we denote by 100. The symbol 346, therefore signifies six single units, 4 packets of ten each, and 3 packets of a hundred each. If there are as many as ten sets of 100, they are to be specified by 1000, and so on, as ordinarily learnt.

This system of notation extends as far as we like to the left of the units place, and if six empty boxes follow the digit 1, it means a million. But we might suppose boxes added to the right of the units place; can we find any use for them? Let us mark the unit box by a double line nearly round it, so that in a long row



there need be no hesitation about which is the unit box; then put the digit unity into each box. In the unit box it means *one* of some thing, in the next box on the left it means one packet of ten of those same things, and so on; each digit to the left having ten times the value of the one immediately on the right. If this convention were extended to the box on the right hand of the units place, the 1 there would signify the tenth part of a unit, and a 1 in the next box on the right would signify the tenth of a tenth, and so on.

For we know that not only can we group things together into an aggregate, it is possible also to cut them up, or split them into fractions.

Thus the things counted may be bags of money, and each bag may be known to contain or to be worth 100 sovereigns. In that case the figure 6 *might* signify six bags, and so stand for 600 sovereigns. And each sovereign might be called a fraction of the contents of a bag, viz., a hundredth part. But in some of the bags the value might be made up with ten pound notes, and each of them would likewise be fractions of the contents of a bag, viz., the tenth part.

Three such notes may therefore be specified either as

3· ten pound notes
or 30· pounds value
or three tenths or ·3 of the value of a bag;

the stop or mark or point being introduced whenever it is necessary for clearness. Any mark will do. In foreign countries a comma is commonly used, whereas we use a dot placed about the middle of the figure. In early days a |‾‾ mark of this kind was used. Thus 346|57 used to be written where we should now write 346·57, or a Frenchman 346,57, the digits 5 7 being partitioned off to signify that they represent fractional parts of objects or units; the digit 6 refers to whole objects or units, the digit 4 to packets of ten, the digit 5 to fractions of one-tenth, and the digit 7 to one-tenth part of tenths, that is to say, it signifies seven hundredths of a unit.

Suppose, for instance, the unit was a bag of sovereigns, as above specified, then the number written 346|57 or 346|57 or 346·57 would mean 346 complete bags of a hundred pounds each, with 5 ten pound notes and 7 sovereigns loose. The money specified would be equal in value to

3465·7 ten pound notes
or to 34657· sovereigns
or to 34·657 thousand pound notes
or to ·034657 million pound notes,

the position of the figures being changed according to the unit intended, and the dot or other mark being used to signify where whole numbers end and fractions begin.

The position of the above numbers relative to each other is constant, viz. the order 3, 4, 6, 5, 7; but their absolute position, or position relative to the unit place, is different in the different cases, and is specified by the dot, which is *always* and invariably placed after the units digit whenever it is inserted at all. It is not always necessary to insert it. For instance the number 3 might be written more completely and equally well 3· or 3·0 or 3·000, in which case it definitely signifies 3 units of something, and the 0 would indicate the fact that there was no fraction to be attended to. If the dot is placed thus 30·, it would mean 3 packets of ten units; if placed thus 300·, it means three groups of ten packets each; and any digit placed after the dot thus ·3 means a fraction, viz. three-tenths of a unit. Whereas if a digit occurs 2 places to the right of the dot, as ·03, it means three hundredths of a unit; as for instance 3 sovereigns would be 3 hundredths or ·03 of a bag in the above example, or ·3 of a ten-pound note. Similarly a florin is one-tenth of a pound or £0·1. Again it is the hundredth, or ·01, of a ten-pound note.

This use of the dot is only a matter of nomenclature, and its importance lies in its simplicity and convenience. It is always possible to write ·03 as $\frac{3}{100}$ if we please, just as it is possible to denote 1864 by MDCCCLXIV if we like; but it is not so simple.

It may be as well to observe that although there is no numerical difference between 6 feet and 6·00 feet, there is a practical and convenient difference of signification. In practice 6 feet would mean something approximately the height of a man, whereas 6·00 feet would be understood to signify either that you had measured a length accurately to the hundredth of a foot or something like the tenth of an inch, and found no fraction; or else that you wished something to be made to that amount of accuracy.

Another way of reading the symbol ·03 is three per cent., or three divided by one hundred. So also five per cent. is ·05; twenty per cent. is ·20; seventy-four per cent. is ·74, and so on.

In the case of twenty per cent. it may obviously be written ·2 or $\frac{2}{10}$ or $\frac{1}{5}$. So also ·5 being 5-tenths or 50 per cent. is the same as $\frac{1}{2}$; and one-half is often the neatest way of speaking of it and writing it. Again twenty-five per cent., or ·25, is the same as $\frac{1}{4}$, being 25-hundredths; and ·125 or 125-thousandths is the same thing as $\frac{1}{8}$. Sometimes one specification is handiest, sometimes the other.

Unfortunately it is not very easy to denote either $\frac{1}{3}$ or $\frac{1}{6}$ or $\frac{2}{3}$ in any other very convenient way on our decimal system of notation, as it would have been if we had arranged to reckon in dozens.

One-third of 1/6 is easy enough, being sixpence, while two-thirds is 1/0: but one-third of 16 is an inconvenient number to write in the ordinary notation. It is $\frac{16}{3}$, that is 16 divided by 3, that is 5·333333... without end, as you find by simple division.

So also $\frac{2}{3}$ of 16 is 10·6666....

These are called repeating or circulating decimals, and their frequent occurrence in ordinary transactions is caused by our unfortunate custom of reckoning in tens instead of in dozens. A simple circulating decimal may always be interpreted as so many ninths: thus whereas ·3 means 3 tenths, ·333... means 3 ninths, which is the equivalent of one-third; ·6666... means 6 ninths, and so on.

A third	of ten is 3·333...
A sixth	of ten is 1·666...
Two-thirds	of ten is 6·666...

and even other fractions are not very convenient.

Thus	a quarter	of ten is 2·5
	an eighth	of ten is 1·25
	a sixteenth	of ten is ·625
	three-quarters	of ten is 7·5

and the only simple things to specify are $\frac{1}{5}$ of ten, which is not often wanted, viz. 2, and a half of ten, which is 5.

This may be contrasted with the convenience of reckoning in dozens:

a third	of a dozen	is 4
a sixth	of a dozen	is 2
two-thirds	of a dozen	is 8
a quarter	of a dozen	is 3
half a dozen		is 6
three-quarters	of a dozen	is 9
an eighth	of a dozen	is $1\frac{1}{2}$
a sixteenth	of a dozen	is $\frac{3}{4}$.

Circulating decimals would not be *avoided* by the duodecimal notation, but they would be rarer, for they would then in the simplest possible cases signify fifths or sevenths or elevenths, which are not the commonest fractions to come across in practice.

It should be remarked that in actual practice circulating decimals only occur in the translation of numerical fractions; and then the decimals *always* either terminate or recur: but in real concrete measurement, or subdivision of continuous magnitude, circulating decimals *never* occur, because such a specification would signify an infinite accuracy, which is impossible.

In all practical cases measurements can only be accurate to a certain number of significant figures, and though it may once in a lifetime happen that these figures are all the same by accident—as for instance 4·4444—it cannot matter in the end whether the last figure is 3 or 5 or even some other digit. When the figures have expressed the actually attained accuracy, all subsequent ones are superfluous and even misleading, because they pretend to an amount of accuracy not really attained.

For this reason the doctrine of circulating decimals belongs rather to pure than to applied mathematics.

In the duodecimal system the ordinary fractions would be denoted as follows :

$$\tfrac{1}{2} = \cdot 6$$
$$\tfrac{1}{3} = \cdot 4$$
$$\tfrac{1}{4} = \cdot 3$$
$$\tfrac{1}{5} = \cdot \dot{2}49\dot{7} \text{ or approximately } \cdot 25$$
$$\tfrac{1}{6} = \cdot 2$$
$$\tfrac{1}{7} = \cdot \dot{1}86t3\dot{5}$$
$$\tfrac{1}{8} = \cdot 16$$
$$\tfrac{1}{9} = \cdot 14$$
$$\frac{1}{\text{ten}} = \cdot 1\dot{2}49\dot{7} \text{ or approximately } \cdot 125$$
$$\frac{1}{\text{eleven}} = \cdot \dot{1}11111$$
$$\frac{1}{\text{twelve}} = \cdot 1$$

Once we have realised the advantages of what is known as the duodecimal system, it is painful to have to return and use the decimal notation.

Nevertheless a change from one to the other would necessitate the uprooting of too deep-seated traditions. Among other things it would alter the multiplication table, that necessary but laborious thing to learn In teaching children it should be realised by the teacher that the multiplication table is hard and tedious, and too much should not be expected of them ; but for convenience of life it is one of those things that it is best to know thoroughly, and it is useful as a matter of discipline. Its rational basis should be understood, and experiment should be encouraged in the first instance to find out what, say, four sixes or seven nines are. It is fairly easy to see that four sixes will make two dozen, it is not so easy to see that they will make two packets of ten and four over, but, the fact having been ascertained, it should be learnt that four sixes are 24, or four times six are 24—either way, whichever happens to be asked, but not both ways at the same time so as to spoil the rhythm.

Similarly it can be ascertained that five sixpences amount to half-a-crown or 2/6; but that five sixes are 30, that is they just make three packets of ten.

It is a serious addition to the work of childhood in this country that they have to learn virtually two distinct multiplication tables, viz. the duodecimal pence table and the decimal or ordinary numerical table. There is plenty of scope for discipline in these things, and so if it is possible to relieve the tedium in other places it is permissible.

The extent of multiplication table to be learnt is merely a matter of convenience, and it is handy to learn beyond 12 times 12. Especially is it convenient to remember that

$13 \times 13 = 169$	$17 \times 17 = 289$
$14 \times 14 = 196$	$18 \times 18 = 324$
$15 \times 15 = 225$	$19 \times 19 = 361$
$16 \times 16 = 256$	$20 \times 20 = 400$

Also that $9 \times 16 = 12 \times 12 = 144 = 1$ gross.

[The square numbers may with advantage be specially emphasised; 1, 4, 9, 16, 25, 36, and so on; and it is easy also as an exercise to ascertain and remember the powers of 2, especially that 32 is the fifth power of 2: they are

2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, etc.,

the last written being the 10th power.

A few of the powers of 3 are also handy.

3, 9, 27, 81, 243, 729.

The cubes or third powers of the simple numbers are useful.

$1 \times 1 \times 1 = 1$	Cube of 7 = 343
$2 \times 2 \times 2 = 8$	„ 8 = 512
$3 \times 3 \times 3 = 27$	„ 9 = 729
Cube of 4 = 64	„ 10 = 1000
„ 5 = 125	„ 11 = 1331
„ 6 = 216	„ 12 = 1728

All this is to be arrived at merely by simple multiplication, and the phrase cube number need not yet be used.]

CHAPTER III.

Further consideration of Division, and introduction of Vulgar Fractions.

JUST as Multiplication is cumulative addition, so Division may be regarded as cumulative subtraction. Thus, for instance, when we say that 7 will go in 56 eight times, we mean that it can be subtracted from 56 eight times. From 59 it can likewise be subtracted eight times, but there will be 3 over. This is the meaning of remainders.

To divide £748. 6s. 11d. by £320. 2s. 4d. we can proceed if we like by subtraction—it happens indeed to be the easiest way,—and having subtracted it twice, we find that that is all we can do, and that there is £108. 2s. 3d. over. So we say that the smaller sum goes twice in the bigger one, and leaves a certain remainder.

In general however it is more customary to regard division as the inverse of multiplication; and, so regarded, it leads straight to fractions and to factors. Thus the fact that 3 multiplied by 4 equals 12, $(3 \times 4 = 12)$, may be equally well expressed by saying that 12 divided by 3 equals 4, $\left(\frac{12}{3} = 4\right)$, or that 12 divided by 4 equals 3, $\left(\frac{12}{4} = 3\right)$, or that 3 and 4 are corresponding factors of 12. Similarly 2 and 6 are other corresponding factors, since $12 \div 6 = 2$ and $12 \div 2 = 6$.

A number like 144, or one gross, has a large number of factors. It is a good easy problem-exercise to suggest to a child to find them all. They are 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72. The factors of 1728 are of course still more numerous. And even the number 60 has a fair number of factors, viz. 2, 3, 4, 5, 6, 10, 12, 15, 20, 30. These may be contrasted with the poor show of factors exhibited by 100, viz. 2, 4, 5, 10, 20, 25, 50.

Children can readily be set to find the factors of numbers, and will thus incidentally be doing many simple division sums.

Their attention must not however be too exclusively, *i.e.* for too long together, directed to integer or whole number factors; they must be prepared to write down the result of division when it is not a whole number, but a fraction, or a whole number plus a fraction. Thus $\frac{144}{5}$ for instance will be found to be 28 and four over, the meaning of which should be carefully explained, being first thoroughly understood and led up to by the teacher.

To lead up to it, it may be pointed out that just as

28 oranges = 20 oranges + 8 oranges
so 28 half oranges = 20 half oranges + 8 half oranges
and 28 halves = 20 halves + 8 halves
and 28 quarters = 20 quarters + 8 quarters; just as much
as 28 farthings = 20 farthings + 8 farthings.

$$\text{Now } \frac{28}{2} = 14, \text{ while } \frac{20}{2} + \frac{8}{2} = 10 + 4 = 14,$$

$$\frac{28}{4} = 7, \text{ while } \frac{20}{4} + \frac{8}{4} = 5 + 2 = 7,$$

$$\frac{28}{10} = \frac{20}{10} + \frac{8}{10} = 2 + {\cdot}8 = 2{\cdot}8,$$

$$\frac{28}{5} = \frac{20}{5} + \frac{8}{5} = 4 + \frac{8}{5};$$

but it is neater to write it

$$= \frac{25}{5} + \frac{3}{5} = 5 + \frac{3}{5} = 5 + \frac{6}{10} = 5{\cdot}6.$$

So now the child should realise that, since $144 = 140 + 4$, so $\frac{144}{5} = \frac{140}{5} + \frac{4}{5}$; which indicates a division that can be done and a division that cannot be done. The division that can be done has the result 28; the division that cannot be done is $4 \div 5$, and it must be left, either in the form of $\frac{4}{5}$, or in the form $\frac{8}{10}$ or ·8. So the whole result is expressible as 28·8.

Accordingly a better way of saying that $\frac{144}{5}$ is 28 and four over, is to say that it equals $28 + \frac{4}{5}$, or 28·8.

To get it in the latter form directly and easily, the original 144 should be written 144·0, and then the sum will run

$$5\underline{|144{\cdot}0}\quad\text{quite naturally.}$$
$$\quad 28{\cdot}8$$

Take another example, because the mind of a child is often sadly fogged about this elementary and important matter.

$$\frac{31}{3} = 10 + \frac{1}{3} = \frac{31{\cdot}000\ldots}{3} = 10{\cdot}333\ldots,$$

a result found by simple division, a process which in this case shows not the slightest sign of terminating but goes on for ever.

Again $\frac{15}{2} = 7$ and 1 over, $= 7 + \frac{1}{2}$, or as it is usually written $7\frac{1}{2}$. But in thus writing it the question should occur, How then would one write $7 \times \frac{1}{2}$? and why does not $7\frac{1}{2}$ mean seven halves, or seven multiplied by a half, or $3\frac{1}{2}$? It is a mere convention, and not a consistent one, that $7\frac{1}{2}$ shall signify $7 + \frac{1}{2}$ and not $7 \times \frac{1}{2}$, and some confusion is thereby caused. By no means need the practice be altered: children must learn to accommodate themselves to existing practice, and must begin reform later in life if ever; but the teacher should realise that the simplicity of $7\frac{1}{2}$ to him is only because he has got accustomed to it, that it is a confusing thing in reality, and that a child who is confused by it is likely to be the bright child and not the dull one.

Expression of vulgar fractions as decimals.

There is nothing new to be learnt about expressing a vulgar fraction in the decimal notation, it is only a question of practice. It is probable that beginners will find no difficulty, but will simply divide out. If any difficulty is felt it can be met by some such initial treatment as the following:

$$\frac{1}{2} \text{ is the same as } \frac{2}{4} \text{ or } \frac{3}{6} \text{ or } \frac{4}{8} \text{ or } \frac{5}{10},$$

and each one of these may therefore be written ·5, which means 5 things in the tenths place or compartment devoted to tenths.

A florin for instance is the tenth part of the value of a sovereign, so 5 florins $= \frac{1}{2}$ a sovereign. £7·5 means 7 pounds + 5 florins or £7. 10s. or £$7\frac{1}{2}$.

So also $$\frac{1}{4} = \frac{2}{8} = \frac{3}{12} = \frac{2\frac{1}{2}}{10}, \text{ etc.},$$

so to express $\frac{1}{4}$ in decimals we shall have to put $2\frac{1}{2}$ in the tenths place; but it is not customary to place fractions there, the $\frac{1}{2}$ is best set down as 5 in the next place to the right, as ·25. In that place 5 will mean $\frac{5}{100}$ths, and that is the same thing as $\frac{1}{2}$ a tenth, viz. $\frac{1}{20}$th.

So $\frac{1}{4}$ of a ten pound note = £2. 10s. = £$2\frac{1}{2}$

= £2·5 = ·25 ten pound note,

and generally $\frac{1}{4} = \cdot25.$

So also $\frac{3}{4} = \cdot75$, $\frac{1}{8} = \cdot125$, etc.

The expression of any fraction as a decimal involves nothing more than simple division; thus $\frac{3}{7}$ can be written ready for operating $7\,|\,\underline{3{\cdot}00000}$, and the quotient, written below, will be ·42857 etc.

In this particular instance however there happens to be no simplification, so the operation is hardly worth performing in that case.

To prove that $\frac{345 \cdot 42}{6} = 57 \cdot 57$

work thus:

$$6 \,\underline{|\, 345 \cdot 42}$$
$$57 \cdot 57.$$

To find $\frac{3475}{8}$ we do the simple division sum

$$8 \,\underline{|\, 3475 \cdot 000}$$
$$434 \cdot 375.$$

Hence $\frac{34 \cdot 75}{8} = 4 \cdot 34375$; $\frac{3 \cdot 475}{8} = \cdot 434375$, etc.

It is not really necessary to write it out in the division form: simple division can be performed on the fraction as it stands.

In every case of writing decimal numbers one under the other, the rule is to keep the column of decimal points vertical; in other words, adhere to your system as to which is the units place, which the tens place, and which the tenths, etc., throughout

Extension of the term multiplication to fractions.

The ordinary idea of multiplication involves the repetition of the same thing several times, as three times four, or seven nines.

The adding of seven nines together is what is called multiplying nine by seven.

The payment of four £5 notes is not called multiplying £5 by 4: but if a conjuror extracted ten apples out of a hat into which one had been put, he might be said to have multiplied it.

So also seed corn is multiplied into an ear; and thus the notion of increase is associated with the notion of multiplying. But it is best to dissociate the notion of increase from the notion of technical multiplication, and to be prepared to multiply by 1 if need be, leaving it the same as before, or even by $\frac{1}{2}$, leaving it smaller than before. This phrase "multiply by a half" is not a simple and natural one: it is a permissible extension, such as we constantly make in mathematics, when any operation that has been found practically useful is applied over the whole range within which it is possible, and sometimes a long way beyond where it appears possible at first sight.

Multiplication by $\frac{1}{2}$ has some points in common with the addition of a negative quantity; it results in diminution, and it is a process that would not have occurred to us to do except as an extension of a straightforward process. To multiply by $\frac{1}{2}$ and to divide by 2 is precisely the same thing. Why not call it then dividing by 2? Well, we do very often, but not always, and a beginner must be content to be told that it is useful to extend the nomenclature of operations in this way. We shall speak of multiplication by $\frac{1}{3}$ if we choose, when we mean division by 3. We shall occasionally speak of adding -4 to a number when we really mean taking 4 from it. We shall do any of these things when we have good reason for doing so, and not otherwise.

Suppose we say that $2\frac{1}{2}$ sovereigns are equivalent to 50s., we arrive at the result by multiplying 20 by $2\frac{1}{2}$, that is first by 2 and then by $\frac{1}{2}$, and adding the separate results. It would be a nuisance to be obliged to say that we multiply 20 by 2 and divide 20 by 2 and add the results, though it would be quite true.

The fact that the half of 20 is 10 may be written if we like, thus: $\frac{1}{2} \times 20 = 10$; or, of course, $\frac{20}{2}$ or $20 \div 2 = 10$;

$\frac{1}{3}$ of $24 = 8$ may be written $\frac{1}{3} \times 24 = \frac{24}{3} = 8$;

$\frac{1}{2}$ of $\frac{1}{3}$ may be written $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$;

and that the half of the third of an apple or ribbon is a sixth of the apple or ribbon is easily verified by experiment. An experiment need not always be performed; after a time it can be vividly imagined, with advantages on the side of clearness of apprehension.

The natural word to use for taking the fraction of a thing is the word "of," like the half of an orange or a quarter of a pound or one-sixth of the revenue; and we shall gradually find that in all arithmetical cases the word "of" has to be interpreted as an instruction to perform the operation denoted by ×, that is to say, the operation we have been accustomed to call multiplication.

Practical remarks on the treatment of fractions.

It so happens that the **multiplication** of vulgar fractions is easier than addition and subtraction, and so it may take precedence. One half of one quarter is one eighth: as can be found by concrete experiment, for instance on an apple, or by looking at the divisions on a 2-foot rule.

$$\tfrac{1}{2} \text{ of } \tfrac{1}{4} = \tfrac{1}{8} = \tfrac{1}{2} \times \tfrac{1}{4}$$
$$\tfrac{1}{2} \text{ of } \tfrac{1}{3} = \tfrac{1}{6}$$
$$\tfrac{1}{4} \text{ of } \tfrac{3}{4} = \tfrac{3}{16}$$
$$\tfrac{1}{4} \text{ of } \tfrac{1}{8} = \tfrac{1}{32}\,;\; \tfrac{1}{4} \text{ of } \tfrac{7}{8} = \tfrac{7}{32}\,;\; \tfrac{3}{4} \text{ of } \tfrac{7}{8} = \tfrac{21}{32}.$$

Such a statement as the last must be, and is, led up to; and gradually the empirical rule can be perceived, that in multiplication of fractions the numerators must be multiplied for the new numerator, and the denominators must be multiplied for the new denominator.

[But initial difficulties and confusion must be expected between this and the **addition** of fractions. Thus, for instance:

$$\frac{3}{4}+\frac{7}{8} = \frac{24+28}{32} = \frac{52}{32}.$$

This is set down here as a warning.

The greatest difficulty in dealing with fractions is felt as long as they are abstract. "$\tfrac{3}{4}$ of what?" is constantly or should constantly be asked by a child. In the above two sums the answer to this question would be different :—

In one it is $\frac{3}{4}$ of a fraction, viz. $\frac{3}{4}$ of $\frac{7}{8}$ of a unit, such as a foot, that has to be found. In the other it is $\frac{3}{4}$ of one foot which has to be added to $\frac{7}{8}$ of another.]

It is convenient to ascertain and remember that $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$ [whereas $\frac{1}{3}$ of $\frac{1}{6}=\frac{1}{18}$]; also that $\frac{1}{4}+\frac{1}{12}=\frac{1}{3}$, or $\frac{1}{3}-\frac{1}{12}=\frac{1}{4}$.

Exercise.

Find the third plus half the third of eight. The answer is 4, but the decimal notation confuses the matter:

$\frac{1}{3}$ of 8 is 2·6666... and half this is 1·333...

so the sum is 3·9999..., that is $3\frac{9}{9}$ or 4.

So also a third + half a third of ten would seem troublesome, though it results simply in five. But a third + half a third of a dozen is simple enough, being $4+2=6$. And always $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$.

Division of fractions may be exhibited thus:

Suppose we have to find what $\frac{7}{8}\div\frac{3}{5}$ amounts to,

write it thus, $\dfrac{\text{seven eighths}}{\text{three fifths}}$

$$=\frac{\text{seven}\times\text{five fortieths}}{\text{three}\times\text{eight fortieths}}=\frac{7\times5}{3\times8}=\frac{35}{24}=\frac{7}{8}\times\frac{5}{3};$$

wherefore instead of dividing by $\frac{3}{5}$, we find we may multiply by $\frac{5}{3}$.

The idea underlying the above process is that things called eighths have to be divided by things called fifths, and that to make it possible they must be expressed in the same denomination, which in this case is fortieths. Thus we get the rule, invert the divisor and multiply. Or otherwise expressed: to divide by a number multiply by its reciprocal. Division by $\frac{1}{2}$ is the same thing as multiplication by 2. The symbol $\div\frac{1}{4}$ is equivalent to the symbol $\times 4$.

CHAPTER IV.

Further consideration and extension of the idea of subtraction.

If a man gains £21. 6s. 5d. and loses £15. 4s. 4d., his nett gain is found by subtraction, and is called the "difference," viz. £6. 2s. 1d.; the total money which has changed hands being the "sum," viz. £36. 10s. 9d. A loss may be called a negative gain; thus a gain of £10 minus £6, would mean a gain of £10 accompanied by a loss of £6, or a nett positive gain of £4. This leads us to discriminate between positive and negative quantities, and to regard subtraction as negative addition. Subtracting a positive quantity is the same as adding an equal negative one.

Geometrically it is sometimes convenient to discriminate between the journey A to B, or AB, and the journey B to A, or BA, just as a French-English dictionary is not the same as an English-French dictionary. When expressed numerically a length AB may be denoted by its value, say 3 inches, or 3 miles; and the reverse journey may be denoted by -3 inches or -3 miles, because this when added to, or performed subsequently to, the direct journey, will neutralise it and leave the traveller where he started. The two opposite signs cancel each other in this sense, and the two quantities added together are said to amount to zero algebraically—that is when their signs are attended to, and as regards the end result only; but the traveller will himself be conscious that although he is where he started from, he has really walked 6 miles; so that for some purposes such quantities may be added, and they are then said to be "arithmetically" or better "numerically" added; for other purposes they are to be "numerically subtracted," or, as it is called, "algebraically added," that is with their signs attended to, and with "minus" neutralising an equal "plus."

If a height above sea level is reckoned positive, a depth below may be reckoned negative; so that a well may be spoken of either as 60 feet below or as -60 feet above the sea level.

The latter mode of specification sounds absurd, but one should gradually accustom one's self to it, for practical purposes later on.

If children feel a difficulty with these negative quantities, as they have every right to, they can be accustomed to them gently, as a horse to a motor car. Mathematicians found some difficulty with them once upon a time, so the difficulty is real, though like so many others it rapidly disappears by custom. Debts, return journeys, fall of thrown-up stones, losings, apparent weights of balloons or of corks under water, dates of reckoning B.C., and many other things will serve as illustrations; not, however, to be taken all at once.

Time is the one thing that never goes backwards; but nevertheless intervals of time may be considered negative if they date back to a period antecedent to the *era of reckoning*. In a race, for instance, it would be an ordinary handicap if one of the competitors was set 12 yards behind scratch, or if he was made to start from scratch 3 seconds late. In either case he could be said to have a negative start.

In golf handicaps it is customary to denote these positions behind scratch as **positive**, because they are added to the score. This is because the object in golf is to get as low a score as possible, not a high one as at cricket.

Addition and subtraction of negative quantities.

Suppose a man inherited a lot of debts, his property would be diminished by their acquisition. The addition to it would be negative, and would be indistinguishable from subtraction.

A debt of £300 added to a possession of £500 would result in nett property of £200; which we might express by saying that $-3+5=+2$.

Or of course the debt might exceed the possession and leave a balance of debt. For instance $-8+5=-3$; where the unit intended by these digits might be a hundred or a thousand pounds. This may be taken as an illustration of the gain of a negative quantity. Take another.

An axe-head at the bottom of a river weighs 3 lbs. Some corks, which, when submerged, pull upwards with a force equal to the weight of 49 ounces, are attached to the mass of iron. Its weight is thus more than counteracted, and it is floated upwards with a force equal to the weight of 1 ounce, because $48-49=-1$.

A raisin at the bottom of a champagne glass, or a speck of grit in a soda-water bottle, can often be seen to accumulate bubbles on itself till it floats to the surface and gets rid of some, when it sinks again, and so on alternately.

The negative or upward weight of the corks, or of the bubbles, counteracts and overbalances the positive weight of the iron or of the fruit. It may be said that we have subtracted more weight from it than it itself possessed, and so left it with a negative weight—like a balloon. The weight of a balloon is not really negative, but it superficially appears to be; because the surrounding air buoys it up with a force equal to the weight of the air it displaces, which represents a greater weight than its own.

When we have to subtract a bigger number from a smaller, we must not always merely say we cannot do it. It is convenient in subtraction sums to say so, and to "borrow" from the digit in the next higher place (*i.e.* to undo one of the available packets and bring the contents one step down), so long as there is something there to be "borrowed," but if we perceive that at the end of the sum there will be a manifest deficiency we must proceed differently.

Suppose we were told to collect £8 from a man who had only £3, we could not really do it; but we might report to our chief, "if we do we shall leave him £5 in debt to somebody," which could be expressed arithmetically thus:

$$3 - 8 = -5.$$

Suppose we were told to pull 5 feet of a gate-post out of the ground, and when we came to try we found that it had only 2 feet buried; we might at first say that it could not be done; but on second thoughts we could say that it was hard to do, and that the only plan we could see would be to pull it minus 3 feet out first, that is to get a mallet and drive it 3 extra feet in, before pulling at it at all.

Suppose a stone were 30 feet above the ground, and we were told to drop it 36 feet, that is to subtract 36 feet from its height of 30 feet. It would not be easy to do, but it could be done, for we might dig a hole 6 feet deep; or it might even be sufficient if we dropped it over a pond of that depth. In either case it would afterwards be 6 feet below the surface of the ground, for $30 - 36 = -6$; it would then be at an elevation of -6 feet, which means the same as a depression of 6 feet.

To speak of a depth of 6 feet as a negative height, in ordinary conversation, would be absurd; but to interpret an arithmetical answer, which gives a height as − 6 feet, to mean that a thing is not elevated at all but is depressed 6 feet, would be quite right and in accordance with commonsense.

Hence the following examples are correct:

$$4 - 9 = -5$$
$$17 - 39 = -22$$
$$546 - 827 = -281$$

But now here is a necessary caution. Take the last case. We see that it is right, for if we add 281 to 546 we get 827; but suppose we had put it down like an ordinary subtraction sum and noticed nothing wrong with it, it would have looked like this

$$\begin{array}{r} 546 \\ \underline{827} \\ -\underline{\underline{319}} \end{array}$$ (example of the way *not* to do it).

We should have said in the old-fashioned way 7 from 6 we cannot, so borrow 10 from the next place; 7 from 16 is 9, put it down. Now we have either 2 from 3, or what is more commonly said, and comes to the same thing, 3 from 4, leaving 1, which we put down; and then we have to take 8 from 5. There is nothing more to borrow, so we must set it down as − 3. Well that is not wrong, but it requires interpreting, and it is not convenient. The minus sign only applies to the 3, which, being in the third place, means 300; the other figures, the 19, were positive. Hence the meaning is $-300 + 19$, or in other words − 281. It might be written $\bar{3}19$, with the minus sign above and understood to apply only to the digit 3, but it could not properly be written − 319.

The above is therefore a very troublesome way of arriving at the result. The convenient way is not to begin performing the impossible subtraction, but to perceive the threatening dilemma, and invert it at once; then subtract the smaller number from the bigger in the ordinary way, labelling the result however as negative. This is of course what we really do when we say $5 - 8 = -3$. We do not begin saying "8 from 5 we cannot, so borrow" from nowhere, for there is nowhere to borrow from. We stop, invert the operation, and record the result as negative; because $a - b = -(b - a)$.

One more case we must take however, viz. where the quantity to be subtracted is itself negative: and its subtraction therefore represents a gain. The loss of an undesirable burden was esteemed by Bunyan's Pilgrim to be a clear gain. A negative subtraction is a positive addition.

$$6 - (-3) = 9; \quad 7 - (-9) = 16.$$

This is sometimes expressed by saying that two minuses make a plus. The effect of a minus is always to reverse the sign of any quantity to which it is prefixed, so if applied to a negative quantity it turns it into a positive quantity. It is equivalent to more than the removal, or subtraction, of a debt, which would be effected by an equal sum added. A loss is more than neutralised by a negative sign, it is reversed.

Add -31 to 114, the result is 83; but subtract -31 from 114, and the result is 145.

No more words are necessary. Familiarity and practice will come in due course as we proceed. A surviving puzzle may occasionally be felt, and can from time to time be removed. It is a mistake to hammer at a simple thing like that till it becomes wearisome; for trifling puzzles or fogginesses evaporate during sleep, and in a few years have automatically disappeared, from children properly taught. They continue to trouble too many adults at present.

CHAPTER V.

Generalisation and extension of the ideas of multiplication and division to concrete quantity.

THE idea of multiplication arose as a convenient summary of a special kind of addition, viz. the addition of several things of the same magnitude to each other. Thus four sixes added together, if counted, make 24, and so it is summarised and remembered as 4 sixes are 24, or 4 times $6 = 24$; and 4 and 6 are called 'factors' of 24.

Originally therefore the two factors in multiplication signified, one of them the size of the quantity of which several are to be added together, and the other the number of times it was to be so added.

Thus 3×6, read 3 times 6, meant a summarised addition sum, $6 + 6 + 6$. But if read 6 times 3 it meant the addition sum $3 + 3 + 3 + 3 + 3 + 3$. That the result is the same may be treated as a matter of experience, and may be demonstrated by grouping, but it is not to be regarded as self-evident. Nevertheless the diagram (fig. 2) demonstrates that 3 rows of 6 each is the same as 6 columns of 3 each. And the counting of window panes and postage stamps are illustrations of practically the same thing.

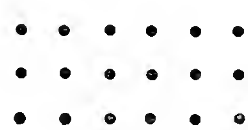


FIG. 2.

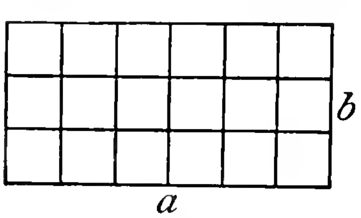


FIG. 3.

Thus we get led to the area of a rectangle of length a and breadth b as $a \times b$, or briefly written ab (fig. 3).

But the idea of multiplication soon generalises itself, and the expression ab gets applied to a number of things to which a simple numerical idea like 3 times 6, or a times b, would hardly apply.

It may be worth showing however that the numerical notion will apply further than might have been anticipated, for instance the rectangle (fig. 4) is built up of 5 equal staves each of them say 3 inches long and an inch wide. The area of each stave is thus 3 inches × 1 inch, or 3 square inches. And by adding 5 of the staves together (or multiplying one of them by 5) we get the total area.

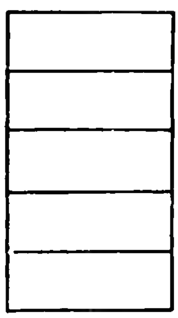


Fig. 4.

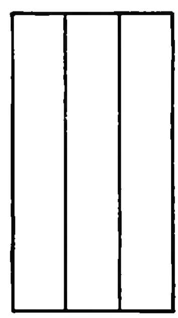


Fig. 5.

And the same area could be equally well obtained by putting together 3 staves each of 5 square inches area (fig. 5).

The number 12 can be resolved into two factors 3 and 4, as is shown by the annexed group which consists of 3 rows of four dots each, or of 4 columns of 3 dots each, proving that 3 times 4 = 4 times 3.

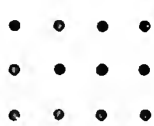


Fig. 6.

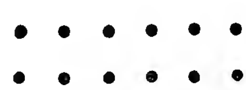


Fig. 7.

A dozen can equally well be grouped as in fig. 7: its large number of factors confers distinction on the number 12.

The number 10 has only two factors, viz. 2 and 5, since the name "factor" is usually limited to whole numbers. It is possible to say that $3\frac{1}{3}$ is a factor of 10, because if it be repeated 3 times the number **ten** results; as is shown by the following set of $3\frac{1}{3}$ disks repeated 3 times, where the central sectors have each of them an angle 120° or $\frac{1}{3}$ of a revolution, and so make up a disk when put together. But the name "factor" is not usually applied to fractions.

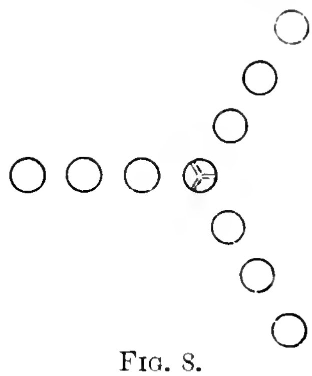


Fig. 8.

Again, a slab of any given area and unit thickness will have a bulk which, measured in cubic inches, is numerically equal to its area in square inches. If such a slab is multiplied or repeated, each slab being piled up on similar ones, say 7 times, then 7 times its bulk will give the volume of a rectangular block; or the volume of a block may be said to be obtained by multiplying its length, breadth, and height. There is no reason to take one of these factors as numerical more than another, and the truth is that none of them need be numerical.

When we say volume = lbh, or length × breadth × height, we may and should mean by l the actual length,

by b ,, ,, breadth,
and by h ,, ,, height,

—*not* the *number* of inches or centimetres in each—and the resulting product is then the actual volume, and not any numerical estimate of it. [If anyone disagrees with this they are asked to withhold their disagreement for the present. This is one of the few things on which presently I wish to dogmatise. See Chap. XXVI. and Appendix II.]

From this point of view the symbols of algebra are concrete or real physical quantities, not symbols for numbers alone, and algebra becomes *more* than generalised arithmetic.

In such cases however the old original definition of multiplication requires generalisation, and a good deal can be written on it; but no difficulty arises, and the question, being interesting chiefly from the philosophic point of view, does not in this book concern us.

We may proceed without compunction to multiply together all sorts of incongruous things if we find any convenience in so doing. Thus, a linear foot multiplied by a linear foot gives a square foot,

$$6 \text{ feet} \times 3 \text{ feet gives } 18 \text{ square feet,}$$

$$4 \text{ feet} \times 3 \text{ feet} \times 2 \text{ feet gives } 24 \text{ cubic feet.}$$

In all these cases something real and intelligible results; but if we multiply square feet by square feet, nothing intelligible results; consequently such a process will never appear in a correct **end**-result, though we shall find that it often appears as a **step in a process** without any detriment.

Again we may multiply a weight by a length, say 3 lbs. by 7 feet, and get what is called 21 foot-lbs., where the unit has a meaning which can be interpreted, viz. the work done in raising a 3 lb. weight 7 feet high against gravity, or else the moment of a force round an axis. But if we try to multiply 3 lbs. by 7 lbs., we should get 21 square lbs., which has no intelligible meaning and is nonsense. There is nothing in the symbols to tell us whether it is sense or not: operations can be consistently performed even on meaningless symbols. To discriminate sense from nonsense, appeal must be made to reality and to actual life and instructed experience.

Division is merely the inverse of multiplication, and similar considerations apply to it.

If we divide 1 by any quantity we get what is called the reciprocal of that quantity.

Thus $\frac{1}{2}$ is the reciprocal of 2. $\frac{1}{10}$ is the reciprocal of 10.

$\frac{1}{3 \text{ feet}}$ is the reciprocal of a length, and could be read

1 per yard.

$\frac{60}{1760 \text{ yards}}$ might represent the number of telegraph posts per mile.

$\frac{1}{\frac{1}{10} \text{ second}}$ is the reciprocal of a time, and might be read 'once every tenth of a second'; or it could be simplified into a repetition of something ten times a second, or 10 per second. It is what is called a 'frequency,' and is in constant use for vibrations.

$\frac{1}{19 \text{ years}}$ is a slow frequency, the frequency with which a cycle of astronomical eclipses approximately recurs.

$\frac{6000 \text{ revolutions}}{5 \text{ minutes}}$ is a frequency of rotation, as of the fly-wheel of a small engine, and may be read as 1200 revolutions per minute, or 20 revolutions per second.

If we divide a length by a time, as for instance $\frac{60 \text{ miles}}{1 \text{ hour}}$, we get a velocity; *e.g.* the speed of an express train.

$\frac{88 \text{ feet}}{1 \text{ second}}$ is exactly the same velocity.

$\frac{4 \text{ miles}}{1 \text{ hour}}$, or approximately $\frac{2 \text{ yards}}{1 \text{ second}}$, is a walking pace.

No hesitation must be felt at thus introducing the units into the numerator or denominator of fractions. If they are left out, the residue becomes a mere numerical fraction, the ratio of two pure numbers; whereas with the units inserted they are real physical quantities with a concrete meaning, and are capable of varied numerical specification.

Thus the velocity of sound in air at the freezing point is

$$\frac{1090\text{ feet}}{1\text{ second}} \text{ or } \frac{33000\text{ centimetres}}{1\text{ second}} \text{ or } \frac{1\text{ mile}}{5\text{ seconds}}$$

$$\text{or } \frac{\cdot 33\text{ kilometres}}{1\text{ second}} \text{ or } \frac{1\text{ kilometre}}{3\text{ seconds}} \text{ approximately}$$

$$\text{or } \frac{10\text{ minutes' walk}}{3\text{ seconds}} \text{ or } \frac{240000\text{ miles}}{\text{a fortnight}}.$$

First idea of involution.

When a number of the same things were added together many times, the process was specially treated and called multiplication. When a number of the things are multiplied together several times, the process is likewise worthy of special treatment, and is called "involution" or the raising of a thing to a certain "power."

The raising to a power is compressed or summarised multiplication. The expression 4×3 meant four **added** to itself 3 times (or 12), whereas 4^3 is understood to mean 4 **multiplied** by itself 3 times (or 64).

So $2^5 = 32$, $6^3 = 216$,

$10^3 = 1000$, $10^6 =$ a million,

$12^2 = 144$, and can be read 12 square, for short; though really a **square number** is an absurdity. It is called "twelve square" because if the 12 represented inches, 12^2 would mean a square foot.

If a is a length, a^2 is truly a square whose side is of length a, and a^3 is truly a cube whose side is of length a. So 4^2 is read "4 square," and 6^3 is often read "six cube," by analogy. It is also true that $2^4 = 16$, but here there is no geometrical analogy, and it is read "2 to the fourth power" simply, the word "power" being often omitted in practice. Similarly a million is "ten to the sixth" or 10^6.

A length divided by a time is a velocity (v), and a velocity divided by a time is an acceleration (a).

$$a = \frac{v}{t}.$$

So in mechanics we find such an expression as

$$s = \frac{1}{2}at^2,$$

where t^2 is often read as the square of the time, although strictly speaking such an expression is nonsense. We can have a square mile, but not a square fortnight; there is no meaning to be attached to the term; time cannot be multiplied by time with any intelligible result. Whenever such an expression occurs, it is to be understood as an abbreviation for something: in the above case for this

$$s = \frac{1}{2}(at)t,$$

where the at is v, and is a real and simple physical quantity.

s is a velocity multiplied by a time, and the double reference to time is caused by the introduction of the specially defined quantity "acceleration," which is often expressed correctly as so many feet per second per second; the two units of time in the denominator being conveniently spoken of as the square of the time—by analogy with geometry again—without thought and without practical detriment, though confusing to anyone who seeks a real philosophic meaning in the expression.

CHAPTER VI.

Factors of simple numbers.

A CHILD should be encouraged who notices that no factor is ever greater than half the number; for though there is nothing in that but what is obvious, yet that is the type of noticing which frequently leads to observations of interest. An even number always has this largest factor, but an odd number can never have a factor greater than a third its value; and frequently its largest factor is less than this. Some numbers have no factors at all; like 7 and 11 and 13 and 29 and 131. These are called **prime numbers**, and a child should make a small list of them as an exercise. But do not attempt to make it learn them or anything of this kind by heart. Ease and quickness of obtaining when wanted is all that is practically needed.

A child should be encouraged to discover criteria for the existence of simple factors; but is hardly likely to be able to notice the facts without aid.

Any number (written in the decimal notation) which is divisible by 3 (*i.e.* which has 3 as a factor) has the sum of its digits also divisible by 3. But this, though convenient as a rule, is in no sense fundamental: it depends merely on our habit of grouping in tens. In the duodecimal system every number ending in 0 would necessarily be divisible by 3 as well as by 4 and by 6; and extremely convenient the fact would be.

For instance, 1/- and 2/- and 4/- and 5/-, or any number of shillings, can be divided by 3, 4, or 6; that is, can be parcelled out exactly into a whole number of pennies.

By reason of the system of reckoning 12 pence to a shilling, any sum of money can be subdivided into three or six equal parts without halfpence or farthings; thus $\frac{1}{3}$ of a pound is 6s. 8d., two-thirds is 13s. 4d., one-sixth is 3/4, one 8th is 2/6, and one-twelfth is 1/8.

In the decimal notation a number has to end in 00 in order to be certainly divisible by 4; and in 000 in order to be certainly divisible by 8. And the division is seldom worth doing even then, because it hardly results in simplification.

The number 5 in the decimal system has an artificial simplicity conferred upon it, but it is not often that we should naturally group things in 5, except for the accident of our 5 fingers: and one of them is a thumb.

The advantage of working in at least two different scales of notation is that it becomes thereby easy to discriminate what is essential and fundamental from what is accidental and dependent on the scale of notation employed. Thus the curious properties of the number nine or eleven are artificial, and in the duodecimal scale are transferred to eleven and thirteen respectively.

The well-known criterion for divisibility by 3 or 9, viz. whether the sum of the digits is so divisible, is accidental again, and disappears in another scale of notation—for instance when units are grouped in dozens instead of tens,—to give place however to a much simpler rule.

The rule about divisibility of the sum of the digits applies to eleven in the duodecimal scale, and indeed would always apply to the number which is one less than the group number artificially selected.

But the existence and identity of **prime numbers** is not accidental at all, but fundamental, and so also is the existence of any given numbers of factors to a number—however it be specified.

Thus one gross can be parcelled out into factors or equal groups in a given number of ways, whether it be denoted by 1/0/0 or by 144 or by any other system of notation.

So also the number one-hundred has only six factors whether it be denoted by 8/4 or by 100 (one nought nought), and its factors are (in the duodecimal scale) :

4/2 2/1 1/8 *t* 5 2,

that is these actual numbers, however they are denoted. In the duodecimal scale it is needful to have single symbols for ten and eleven ; and the initial letters serve the purpose.

An actual number is easily *exhibited* by means of counters or coins or marbles : its expression in digits is an artificial arrangement and is adopted simply for convenience : it is analogous to sorting the marbles into bags of which each must contain an equal number—whatever number may be chosen as suitable and fixed upon for the purpose.

It may be interesting to write down the numbers in the duodecimal scale which would be divisible by 5.

5, *t*, 1/3, 1/8, 2/1, 2/6, 2/*e*, 3/4, 3/9, 4/2, 4/7, 5/0, ...,

and the even numbers in the above are divisible also by ten. The above numbers should be read five, ten, one and three, one and eight, two and one, two and six, two and eleven, etc., meaning one dozen and three, one dozen and eight, two dozen and six, two dozen and eleven, etc.

Numbers which have the factor 7 are

7, 1/2, 1/9, 2/4, 2/*e*, 3/6, 4/1, 4/8, 5/3, 5/*t*, 6/5, 7/0, ...,

and the even ones are divisible also by fourteen.

Numbers which have the factor eleven (*e*) are

e, 1/*t*, 2/9, 3/8, 4/7, 5/6, 6/5, 7/4, 8/3, 9/2, *t*/1, *e*/0, ..., namely eleven, one and ten, two and nine, and so on: the last one written being read eleven dozen.

Numbers divisible by thirteen (1/1) are

1/1, 2/2, 3/3, 4/4, 5/5, 6/6, 7/7, 8/8, 9/9, *t*/*t*, *e*/*e*, 1/1/0, ...

In the last two cases a law or order among the digits is manifest, but in all four cases it may be noticed that **every** digit makes its appearance in the units place, though only in the last two cases do they appear in a simple order.

Numbers divisible by 3 are

3, 6, **9**, 1/0, 1/3, **1/6**, 1/9, 2/0, **2/3**, 2/6, 2/9, **3/0**, ...

and the even ones are divisible by 6. Every third one of the above series, viz. those in thick type, are divisible by 9.

Numbers divisible by 4 are

4, 8, 1/0, **1/4**, 1/8, 2/0, 2/4, **2/8**, 3/0, 3/4, 3/8, **4/0**, ...

Alternate ones are divisible by 8, and those in thick type are divisible by sixteen.

Numbers divisible by twelve, that is arrangeable in dozens, are of course,

1/0, 2/0, 3/0, 4/0, etc., 1/0/0, ...,

the last written being the symbol for a dozen dozen or one gross.

CHAPTER VII.

Dealings with money and with weights and measures.

In the British Isles it is customary to count pennies by the dozen, the value of which when coined in silver is called a "shilling"; and shillings are counted by the score, the value of which is called a "pound sterling," or when coined in gold a "sovereign." Five dozen pence, or a quarter of a pound sterling, when in a single silver piece used also to be called a "crown." And these, together with the half-sovereign, half-crown, half-penny, etc., are the chief names in vogue; except the "guinea" and the "farthing," neither of which need much concern us. The "florin" is an attempt at a decimal coinage, being the tenth of a pound; and the double-florin is an attempt at an international currency or equivalence with the dollar and the five-franc piece.

The addition of money is a practical operation in constant use, and plenty of practice in addition is obtainable by its means. No other addition sums are worth attention for their own sake: but in addition of money it is worth while taking pains to acquire a fairly quick and accurate style. At the same time it is to be remembered that it is a purely mechanical process—one that in large offices is better, more rapidly and more accurately, performed by a machine, into which the figures are introduced by pressing studs, and then the addition performed instantaneously by turning a handle.

Nothing that can be performed by turning a handle can be considered an element in a liberal education: it can only be a practical and useful art. That however it is; partly because a machine is seldom available, partly because it is ignominious to be helpless without a tool of this kind, chiefly because addition of money is an operation which is called for by commonplace daily life more often than any other.

Nothing much need here be said about it. *The columns of an actual account book are the best addition sums to set for practice.* Also, in writing figures down, it is well to take care to place the unit digits under each other, leaving a place for a left-hand digit whenever such occurs in the pence and shillings columns, and to be equally careful to write the pounds with the corresponding places vertical. Also to write all figures very plainly. This last always, and for all purposes: **A good clear style of figure-writing should be cultivated.**

Subtraction of money is greatly facilitated by the use of the "shop" method: the old-fashioned process of "borrowing" was troublesome, and moreover only enabled one row of figures to be subtracted from one row, whereas with the shop or complementary method any number of rows may be subtracted from another row, and the process is practically only addition. For instance suppose it is wished to subtract all the smaller amounts from the larger in the annexed statement:

	£	s.	d.
	341	8	7
less	19	5	9
and	14	0	3
and	36	17	5
	271	5	2

The process is, to say, 5 and 3 and 9 make 1/5 and 2 make 1/7, put down the 2d. and carry 1/-; then 18 and 0 and 5 make 23 and 5 make 28, put down 5/- and carry £1; then

7 and 4 and 9 make 20 and **1** make 21, put down 1 and carry 2; then $5+1+1=7$ and **7** make 14; and finally 1 and **2** make 3. In reading, emphasise all the black figures.

Verify by adding the four lower lines.

As to multiplication and division of money or of weights and measures we will deal briefly with them: the old-fashioned practice in such matters was tedious and was pushed in childhood into needless intricacies. Dulness is apt to line all this region, unless skill is expended on it and due care taken, and no more practice should be enforced in it than is required for ordinary life. Discipline and punishment lessons might possibly with advantage be confined to this region. Even for punishment it is however hardly necessary to inflict sums dealing with acres, furlongs, poles or perches; or with bushels, pecks, scruples, quarters, pennyweights, and drams. Hogsheads, kilderkins, and firkins may perhaps at length be considered extinct, except for purposes connected with the study of folk-lore. There are plenty of real and living units to be learnt in Physics: we need not ransack old libraries and antique country customs for them. And, though the humanity involved in and represented by old names has been a relief to some children, during their dismal lessons, far too much has been made of the trivial and dull operations suggested by tables of British weights and measures. The sooner most of them are consigned to oblivion the better.

Real living arithmetic is the same in any country; and the most important of all is that which must necessarily be the same on any planet.

The units that are at present worthy of terrestrial attention are the following:

Units of length—inch, foot, yard, mile, millimetre, centimetre, metre, kilometre.

units of time—second, minute, hour, day, week year.
units of area—square mile, square foot, square centimetre, etc.
units of volume—cubic foot, cubic inch, cubic yard, cubic centimetre, cubic metre; occasionally also litres, gallons and pints.
units of mass—pound, ton (ounce, grain, hundredweight occasionally), gramme, kilogramme, milligramme.
units of money—pence, shillings, pounds, francs, marks, dollars.

But conversion from one to the other of the last-mentioned denominations should in every case be only approximate. Accurate work when wanted is done by tables, and the rate of exchange is constantly varying.

For division of money, and of weights and measures, the orthodox school rule is called "practice"; and it sometimes happens that by excessive practice children are able to do this kind of sum much better than adults—better even than mathematicians; but since school time is limited, such extravagant facility in one direction is necessarily balanced by extreme deficiency in many others, and is therefore to be deprecated. The world is too full of interest to make it legitimate to exhaust the faculties of children over quite needless arithmetical gymnastics, which confer no mathematical facility, but engender dislike of the whole subject.

Modern treatment of the rule called "Practice." The practical advantages of decimalisation.

In old days some very long sums used to be invented for British children whereby our insular system of coinage and of weights and measures was pressed into the service to make difficult exercises. The form was usually something like this:

Find the cost of 131 tons 5 cwt. 3 qrs. 24 lbs. 5 oz. at £4. 13s. 9½d. a ton.

Mr. Sonnenschein led the way, I think, towards taking all the sting out of these outrageous problems, and reducing them to useful though unimpressive and essentially insular exercises, by introducing the chief advantage of the decimal system into the working, before it had been embodied by Parliament in a legal system of weights and measures and coinage itself.

If such sums have to be done, and a moderate amount of "practice" in that direction is quite legitimate, decimalisation of at least one of the quantities specified, that is, expressing it in terms of one denomination, is undoubtedly the proper initial step to take; and then if we are asked the cost of so much goods at a given price, the matter becomes a mere straightforward multiplication; while if we are asked to find the price of a given amount of goods which have cost so much money, or the amount of goods which can be obtained for a given sum of money at a given price, we have only a straightforward division sum to do;—once the complication of many denominations, that is to say the "compound" nature of the specification, with scales of notation mixed up, is by an initial process got rid of. It is always possible, and sometimes advocated, to reduce everything to the *lowest* denomination, *e.g.*, in the sum above to halfpennies and ounces; but that is terribly long and tedious. Expression in terms of the *highest* denomination is much neater. The initial process is as follows:

Decimalisation of money.

To express any sum of money in terms of a single unit, say £1, which is the best unit for the purpose, it is sufficient to notice and remember a few simple convenient facts. They are all painfully insular, and are not an essential part of real arithmetic at all, but if properly and lightly treated they afford to British children an amount of easy practice which foreign children are destitute of. It is only when trivial facts and insignificant sums are laboured at, till they kill all interest of the British child in real arithmetic, that they become deadly and deserving of the harshest epithets.

The decimalisation of money in terms of a pound is easy, since a florin is the tenth of a sovereign; so any number of shillings is easily expressed in decimals of a pound.

2/- = £·1,	1/- = £·05,
3/- = £·15,	6d. = £·025,
4/- = £·2,	1/6 = £·075,
5/- = £·25,	2/6 = £·125,
6/- = £·3,	3/6 = £·175,
7/- = £·35,	4/6 = £·225,
and so on.	etc.

A penny is $\frac{1}{240}$th of a pound, but that is not specially convenient when expressed as a decimal; a farthing is $\frac{1}{960}$th of a pound, and that is approximately $\frac{1}{1000}$ or £·001.

Since money is never needed closer than to the nearest farthing, except in the price of cotton per lb. and a few rare cases, the approximation of £·001, sometimes called a mil, for 1 farthing, or the writing of a farthing instead of £·001, often suffices; especially in interpreting results.

The following expressions are all equivalent in value:

$\frac{1}{2}$ a sovereign = 10/- = 5 florins = £·5.

So also are the following, each row among themselves:

£7. 10s. = £7$\frac{1}{2}$ = £7·5 = £7 + 5 florins.

$\frac{1}{4}$ of a ten-pound note = ·25 ten-pound note = £2·5 = £2. 10s.

15/- = £$\frac{3}{4}$ = £·75.

150/- = 75 florins = £7·5 = £7. 10s.
18/- = 9 florins = £·9.
12/- = 6 florins = £·6.
£1. 12s. = £1·6.
£4. 18s. = £4·9.
£7. 19s. = £7·95.

All these expressions should be read backwards as well as forwards.

So also

£5. 2s. 6d. = £5·125.
£3. 1s. 6d. = £3·075.
£3. 11s. 6d. = £3·575.
£3. 11s. $6\frac{1}{2}$d. = £3·577, almost exactly.

Take a few examples of the interpretation of decimals of a pound into ordinary coinage:

£1·2 = £1. 4s.
£4·25 = £4. 5s.
£7·904 = £7. 18s. 1d., the four mils being practically a penny.
£13·127 = £13. 2s. $6\frac{1}{2}$d., the ·125 being 2/6, and the 2 extra mils $\frac{1}{2}$d.
£1·178 = £1. 3s. $6\frac{3}{4}$d., the ·15 being 3/-, ·025 = 6d., and there being 3 mils more.

£·025 = 6d.
£·026 = $6\frac{1}{4}$d. almost exactly.
£·027 = $6\frac{1}{2}$d. ,, ,,
£·028 = $6\frac{3}{4}$d. ,, ,,
£·029 = 7d. ,, ,,
£·030 = $7\frac{1}{4}$d. ,, ,, or exactly $7\frac{1}{5}$d.

We are now ready to do any number of sums like the following:

Find the cost of 324 horses at £17. 9s. 6d. a horse.

Now 9/6 = 8/- + 1/6 = £·4 + £·075 = £·475, so the answer is merely

$$324 \times £17{\cdot}475 = £5651{\cdot}9 = £5651.\ 18s.$$

Find the cost of 900 things at £9. 7s. $4\frac{1}{2}$d. (Sonnenschein.)
Answer is £9·3875 × 900 = £8431·875 by simple multiplication
= £8431. 17s. 6d.

How much a year is £31. 9s. 9d. per day?

Answer 365 × £31·4875 = £11492·9375 = £11492. 18s. 9d.

How much interest must be paid for 43 days' loan of a sum of £543. 17s. 6d. at the rate of $3\frac{1}{2}$ per cent. per annum? (Sonnenschein.)

Here £$3\frac{1}{2}$ must be paid for each hundred pounds lent for a year, so for 43 days only $\frac{43}{365}$ths of that sum has to be paid.

Now 17/6 = 8 florins + 1/6 (or, otherwise, seven-eighths of a pound) = £·875; so the amount to be paid is:

$$\frac{3\frac{1}{2}}{100} \times \frac{43}{365} \times £543{\cdot}875;$$

that is to say,

$$\frac{7 \times 43}{730} \times £5{\cdot}43875.$$

This yields £2·243 = £2. 4s. $10\frac{1}{4}$d., the answer.

Typical exercises.

There are certain time-honoured exercises of a type such as the following, in which a fair amount of practice is desirable. [Type only here given.]

If 3 peaches cost a shilling, what will 20 cost?

If I have to pay 15 workmen at 10d. an hour for 8 hours, how much money do I need?

If the butcher supplies $7\frac{1}{2}$ lbs. of meat for 5s., what has he charged per pound?

and so on. The last being a troublesome kind of sum frequently occurring to housekeepers, but usually and most easily done by tables.*

Examples like these are quite harmless and give needful practice, but when they become complicated a little of them is sufficient, except for discipline, and the more concrete and amusing they can be kept for ordinary purposes the better.

A slight further development, not quite so harmless, is of the following type:

Find the cost of 6 lbs. 11 oz. 9 dwt. at 17s. $8\frac{1}{2}$d. per ounce.

In British schools there is far too great a tendency to limit all exercises to pseudo-commercial matters. In real business this kind of sum hardly occurs; and besides, greater interest can be obtained by opening up fresh ground for the subject matter of examples.

A few specimens may be here suggested. A great deal of what has to be laboriously taught later as *physics* is nothing but simple arithmetic, and could easily be assimilated unconsciously while doing sums.

1. If the sound of thunder takes 10 seconds to reach our ears, how far has it come? (See p. 56 for velocity of sound: it travels approximately a mile in five seconds. For more accurate specification the temperature would have to be known.)

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* Answers to these sums are as follows: Each peach costs the third of 1s., so twenty peaches will cost the third of £1, or 6s. 8d.

The 15 workmen's wages will amount to £5, since 80d. is the third of a pound.

The price is 8d. a lb., since $7\frac{1}{2}$d. doubled three times makes 60d. or 5s.

But it will be observed that in each case some accidentally convenient relation is seized and utilised. That is the essence of mental arithmetic: it is a training in quickness and ingenuity, not in mathematics; and its merits can be appraised accordingly.

2. If a pistol shot is heard across an estuary 15 seconds after the pistol was fired (which can be told by observing the flash), how wide is the estuary?

3. If light reaches the earth from the sun in 8 minutes, what is its velocity? (The distance of the sun being 93 million miles.)

4. How long does it take to come from the moon?

5. How long would it take to travel a distance equal to seven times the circumference of the earth?

6. If it takes 5 years to arrive from a star, how far off is that star?

7. If a locomotive could be run 60 miles an hour day and night, how long would it take to go round the earth?

8. How long to reach the sun? etc.*

Answers should be given in weeks or years or whatever unit is appropriate and most suggestive. This is a good rule always, and is the real use of units to which people are accustomed. Conversion of miles into inches is tedious and useless: but stating a big result in miles, a small result in inches, and a moderate result in feet or yards, is right and illuminating.

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* Answers:

1. 2 miles.
2. About 3 miles.
3. 93 million miles ÷ 8 minutes

$$= \frac{93000}{8 \times 60} \text{ thousand miles per second}$$

$$= \frac{3100}{16} = \text{nearly } 194 \text{ thousand miles per second.}$$

4. About a second and a half.
5. About 1 second.
6. $5 \times 365\frac{1}{4} \times 86400 \times 194{,}000$ miles.
7. Nearly 17 days.
8. About 180 years.

9. If a pistol shot fired in a valley, at a spot which is distant from the summit of a mountain by an amount which is represented by a length of 4 inches on an ordnance map of scale 1 inch to the mile, is heard on that mountain top 25 seconds after the flash, how high is the mountain above the valley? (Ans.: 3 miles.)

This is perhaps hard: it can be done by drawing and measuring, after it has been perceived that the sound has travelled 5 miles in a straight line.

10. If a motor car is travelling 21 miles an hour, how long will it take to go 100 yards?

Ans.: 9·74 or $9\frac{3}{4}$ seconds.

11. If the estimate of time were $\frac{3}{4}$ second out, what error would be made in reckoning the speed from the measured distance?

Ans.: $\frac{3}{4}$ sec. is $\frac{1}{13}$th of $\frac{39}{4}$ sec., so the error in estimate of speed would be about

$\frac{21}{13}$ miles an hour, or about $7\frac{1}{2}$ per cent.

12. If a volunteer corps of 84 members shoots 160 rounds a day each for 5 weeks, and if each bullet weighs $\frac{3}{4}$ of an ounce, what weight of lead will they have expended?

13. If each bullet needed one halfpennyworth of powder to propel it, and if lead cost 17/- a cwt., what would be the cost, in powder and shot, for a regiment of 12 such corps, in the course of 5 weeks?

14. If an iron rod expands $\frac{1}{4}$ per cent. of its length when warmed 200 degrees, what allowance must be made for the expansion of a bridge girder, $\frac{1}{8}$ mile long, between a winter temperature of $-40°$ and a summer temperature of $110°$?

15. With the above data how much will an iron rod a foot long expand if warmed one degree?

16. If a snail crawl half an inch each minute, how far will it go in 3 hours?

17. If sound goes a mile in 5 seconds, how long would it take to go a foot?

18. If sound reverberated between two walls 10 feet apart, how many excursions to and fro will it make per second?

19. If light takes 8 minutes to travel 93 million miles, how long would it take to go one yard? How many kilometres would it travel per second? How many centimetres per second?*

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* Answers to the above:

12. $84 \times 160 \times 35 \times \frac{3}{4}$ ounces = about 10 tons.

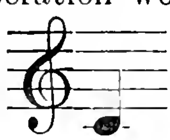
13. In shot, about £170; since a shilling per cwt. is a pound per ton. In powder, $84 \times 160 \times 35 \times \frac{1}{2}$ pence $= 7 \times 80 \times 35$ shillings $= 28 \times 35$ pounds per corps.

14. The range of temperature is 150°; for this range iron expands $\frac{3}{4}$ of $\frac{1}{4}$ per cent. of its length; that is,

$$\frac{3}{16 \times 800} \text{ mile} = \frac{9 \times 1760}{16 \times 800} \text{ feet} = \frac{990}{800} = 1{\cdot}2375 \text{ feet, or nearly 15 inches.}$$

15. $\frac{1}{200}$th of $\frac{1}{4}$ per cent. of its length; which is $\frac{1}{80,000}$th of a foot, or ·0000125 expansion per unit length per degree; which is about the right value for iron.

17. 1 second ÷ 1056, or about the thousandth of a second.

18. In each excursion to and fro it will have to travel 20 feet; but it can travel 1056 feet in a second, therefore it has time to make 51·12 excursions per second. If the walls were only 2 feet apart instead of ten, the rate of reverberation would be 5 times as rapid, and would correspond to the note  This therefore is the musical note heard if a short sharp noise, like a blow or clap, be made between two walls two feet apart.

19. To travel 1 mile, light would take 8 minutes ÷ 93 million; therefore to travel 1 yard it would take 1/1760th part of this.

Ans.: $$\frac{8 \times 60}{93 \times 1760} = \frac{10}{31 \times 110} = \frac{1}{341} \text{ millionths of a second.}$$

Light travels 300,000 kilometres per second or 3×10^{10} centimetres per second; as nearly as experiment at present enables us to say.

20. If you buy a large number of oranges at three a penny, and an equal number at two a penny, and then sell them all at five for twopence, how much have you lost on the transaction?

(Ans.: a penny for every 5 dozen sold.

The buying price per couple is $\frac{1}{3}$d. + $\frac{1}{2}$d.;

the selling price per couple is $\frac{2}{2\frac{1}{2}}$d.

So the loss per couple is $\frac{1}{3}+\frac{1}{2}-\frac{4}{5} = \frac{5}{6}-\frac{4}{5} = \frac{1}{30}$d.)

There are many ways of doing this problem, and it should not be left till it is fully **realised.** Other problems depend on the same principle, which is an important one. For instance:

21. An oarsman rows a boat a certain distance up a river and back, and then across the river, or on a lake, the same distance and back. Which will be the quickest to and fro journey?

22. If a steamer travels down a river at a rate of 19 miles per hour, and up the same river with the same engine-exertion at 7 miles an hour, what is the speed of the river? How long would the steamer take to go a journey of 65 miles and back?

(Ans.: The speed of the boat in stagnant water is **the half-sum**, viz. 13, the speed of the river is the **the half-difference**, viz. 6 miles per hour. The journey of 130 miles would take ten hours in stagnant water, but up and down the river it will take nearly thirteen hours.)

The general principle is that whereas $(1+x)+(1-x) = 2$,

$$\frac{1}{1+x}+\frac{1}{1-x} \text{ does not equal } 2$$

but does equal $\frac{2}{1-x^2}$, which is greater than 2; though not much greater when x is small. This applies to (20) (21) and (22).

23. If a couple of travellers sharing expenses are found to be out of pocket in the course of the day, *A*, £2. 4s. 6d., and

B, £1. 3s. 4d., what sum must be transferred from one to the other to equalise matters?

(Ans.: Half the difference, viz. 10s. 7d.; and the cost to each has been half the sum, viz. £1. 13s. 11d.)

24. If three travellers on a tour have expended when they return

A	£17 . 4 . 6
B	£4 . 3 . 2
C	£7 . 5 . 4

how can they best arrange to share expenses equally?

(*Ans.* Find the mean expenditure by adding the items together and dividing by 3; and then take the difference between this mean and the expenditure of each. *B* and *C* will then have to pay their respective differences to *A*. Their two deficiencies from the mean, added together, should equal *A*'s excess expenditure over the mean; if this is not the case a mistake has been made.)

The same rule would apply to any number of travellers. Observe how it works for the couple of last question.

These exercises do not contain examples of so many quarts, pecks, pennyweights, and drams. Such sums have no business to occur. If artificial complexities of that sort are set, any way of dealing with them will do: the simplest way is the best way.

If a pupil is constrained to bethink himself of how the teacher intends him to do a sum, it destroys originality. His effort should always be devoted to find the best and simplest way. This a teacher can help him to find, but a self-found way is more wholesome in many respects than a coerced way, even though the latter is neater. Originality should always be respected: it is rather rare. Perhaps docility is made too much of, and budding shoots of originality are frozen.

Binary scale.

Although the natural method of dealing with multiples of a unit is to employ the same system of notation as is in vogue in arithmetic, and although therefore it is natural to specify large numbers of things by powers of ten, there is a natural tendency also to deal with **fractions** on a different basis, viz. to proceed by powers of $\frac{1}{2}$. We see this on a foot rule, where the inches are first halved, then quartered, then divided into eighths, then into sixteenths, and sometimes even into thirty-second parts of an inch.

The same method of dealing with fractions is found in prices, as for instance of cotton, or any commodity which requires a penny to be subdivided. Below the halfpenny and the farthing we find the eighth, sixteenth, thirty-second, and sixty-fourth of a penny in use for quotations; and these ungainly figures are, or used to be, even telegraphed and automatically printed on tape. So also a carpenter will understand a specification in sixteenths of an inch, while a decimal subdivision would puzzle him.

A thousandth of an inch is sometimes used however in fine metal fitting work, and the thickness of a rod wanted may be specified to a fitter as the thousandth of an inch greater than $2\frac{3}{16}$ inch.

These peculiarities are insular and not to be encouraged, having originated in laziness and ignorance; but they are not nearly so bad as the weights and measures which people who ought to know better still require that children shall be taught.

It is quite **possible** to word arithmetic itself on the binary scale, counting in pairs only; thus 10 (read one nought) may be understood to mean 1 pair; 100 may mean 1 pair of pairs (or 4), 1000 on the same plan will mean $2 \times 2 \times 2$ or eight, and so on. And on this scale ·1 would mean $\frac{1}{2}$, ·01 a quarter,

·001 an eighth; so that one and a quarter plus an eighth would be written 1·011.

The natural tendency to this kind of subdivision is apparent in coins, even in countries with a decimal currency. For instance in America you find the half and the quarter dollar, beside the dime and the cent. In France you find the double franc, franc, half-franc, and quarter-franc. So in Germany we have as a drink-measure the *halb-liter* and *viertel-liter*. And in England we have half-sovereigns, half-crowns, also three-penny bits, sixpences, shillings, florins, and double florins, each double the preceding; the double florin being roughly equivalent to a dollar or to a five-franc piece.

So also the commonest gold piece in France is the Napoleon or 20-franc piece; not the ten-franc, or the hundred-franc piece, though they both exist.

This natural tendency is the chief difficulty in introducing a purely decimal coinage; another is the convenience of the penny and the shilling. If a decimal system is to be introduced, one or other of these coins must give way. If the shilling gives way, we can have an approximation to the franc, and much inconvenience or grumbling in connexion with cab fares, etc. If the penny gives way, and is made the tenth of a shilling, we approach closely to the German system; and many commodities used by poor people will automatically rise in price.

In Austria an attempt is being made to replace the *gulden* and *kreutzer* by their respective halves, called *krone* and *heller*, which correspond approximately with the franc and centime; but the older denominations persist, and it is quite likely that the two will co-exist and be convenient.

It may be asked "why mention these things in a book of this kind"? And the answer is because children can take an intelligent interest in them, and because it is instructive for them to realise that our present coinage is not a heaven-sent institution, but is susceptible of change,—change too in which, when adult, some of them can take their part, either in promoting or opposing. There is therefore a reality about these things, and arithmetical ideas can inculcate themselves in connexion with them without labour.

Decimal system of weights and measures.

Although the present division of money is so deep-rooted that decimal coinage is difficult of introduction, and although the decimal system in arithmetic is not the best that could have been devised; yet its advantages over most other systems are so enormous that in connexion with weights and measures it undoubtedly ought speedily to be introduced.

The first and easiest place to introduce it is in connexion with **weights.** No one really wants to reckon in ounces and pennyweights and grains and scruples and drachms. Ounces used to be perpetuated and popularised by the Post Office regulations; but now that a quarter of a pound will go for a penny, and, under certain restrictions, an eighth of a pound for a halfpenny, the necessity for ounces has really disappeared. It would be quite easy to make the halfpenny postal regulations refer to a tenth of a pound instead of an eighth, and to construct ten-pound weights, hundred-pound weights, and their convenient doubles and halves and quarters.

There is however this fundamental question to be considered: shall the British pound be adhered to, or shall we adopt the unit of our neighbours and employ the kilo (short for kilogramme) or the demi-kilo?

The kilo is too big for many ordinary purposes. In France small marketing is still done by the demi-kilo, because it represents a reasonable and commonly-needed amount of stuff.

It is altogether handier than the kilo. A demi-kilo might be introduced, and with us might still be called a pound, or, for a time, an "imperial pound," though its value would have to be increased by ten per cent. above our present pound. The kilo is approximately 2·2 lbs., so the new pound or demi-kilo would be one and a tenth old pounds. The gramme would be ·002 new pound.

The disadvantages of any change are obvious. The advantage would be that we should then be using practically the same unit as our neighbours.

All other denominations could be swept away; except, for occasional rough use, the ounce and the ton, which continue useful; for the ton would be 2000 of the new pounds, and would correspond exactly with the French tonne; and the ounce, slightly changed, would be $\frac{1}{16}$ of the new pound, or it might be changed so as to be one-tenth of it. The grain or $\frac{1}{7000}$ part of the old pound might easily give place to a new grain $\frac{1}{10000}$ part of the new one.

These handy names are useful for common purposes and for speech. All accurate specifications should be made in terms of the pound, and of that alone. Thus 1·4903 lbs. would be a specification accurate to the nearest grain of a weighing of something like a pound and a half.

3·014 tons would be a statement, intended to be accurate to the nearest pound, of the weighing of a 3-ton mass.

Let me emphasise what may be regarded as one of the special advantages of this simple and easily introduced change. Children could then be practised in weighing at once: to the vast advantage of their education. At present an apothecary's scales are an abomination, and no child can weigh satisfactorily with the weights of a letter balance, which are all in the binary scale; though, as aforesaid, these serve as an introduction to *ideas* of weighing, etc., in quite early stages. Letter weights go down too rapidly; there are not enough subdivisions; and the result cannot easily and quickly be specified, except as an awkward series of vulgar fractions, or else in the binary scale of arithmetical notation.

The only way in which school weighings can be satisfactorily done now is by the use of grammes and kilogrammes: and there is a foreign feel about these things; which those who learn chemistry indeed get over, but which gives it a flavour distinct from ordinary life.

What we want is that children shall weigh and measure all sorts of things, and do a large part of their arithmetic in terms of their own weighings and measurings: thus making it real and concrete and if possible interesting.

Weighings of plants and of growing seeds, of rusting iron and of burning candles, of dissolving salts and of evaporating liquids, can all be made interesting and instructive.

Weighings in air and water, and finding thereby the specific gravity or the volume of irregular solids, can easily be overdone and made tedious, but, short of this, such operations are quite instructive.

Gauging and measuring of regular solids is an equally instructive way of arriving at their specific gravity, or, as it may be more scientifically called, "density." The approximate relative densities of such things as stone, lead, iron, gold, copper, platinum, cork, air, referred to water, are worth remembering: stone say 2·5, lead 11, iron 7, gold 19, copper 8, platinum 21, cork $\frac{1}{5}$, air $\frac{1}{800}$.

Decimal measures.—*Continued.*

The introduction into commerce of "the decimal system" is a more difficult matter however. The admirable duodecimal division of the foot into inches (like that of the shilling into pence) stands in the way. The foot and the inch and the yard seem ingrained in the British character, and will give place to the metre and the centimetre only with difficulty.

The fact is that the introducers of the "metre" made a great mistake by not adopting the yard or the foot or some other existing unit as its value: they would also have been wise if they had adopted the pound as their kilogramme, and left the dimensions of the earth alone. It is the magnitude of the human body which really and scientifically specifies and confers any meaning on absolute size: our bodily dimensions and time relations must be the basis of all our measures and ideas of absolute magnitude. To abandon the human body and to attend to the dimensions of the earth was essentially unscientific or unphilosophical: it has all the marks of faddism and self-opinionatedness. However these unwisdoms of sections of the human race we have to put up with, and at any rate the French evolved a better system on the whole than that which had come down to us by inheritance and tradition from uncivilised times.

If we were at liberty to adopt the foot as our standard, and to call its decimal subdivisions inches, or if a new foot were made ten inches long, the change would not be so very difficult. If it had been extensively customary to divide the inch too into twelfths (called lines) the change would be harder; but divisions of the inch in the binary scale have been customary, and these are not really convenient: a decimal system is better than that; and foot rules decimally divided and subdivided could easily be supplied and used.

But then, as in the case of our present pound, we should be using an insular measure different from all the rest of Europe, and amid the stress of industrial and engineering competition this is a serious handicap.

A metre scale is a rather unwieldy thing: a half-metre scale is handier for many purposes, and might be made like a folding two-foot rule.

There is no help for it: we must get used to metres and centimetres, and the sooner we begin the better.

Angles and Time.

There are two things which have not yet been subdivided decimally with any considerable consensus of agreement: they are Angles and Time.

The division of the right angle into 90 equal parts is convenient. The subdivision of the degree into sixtieths and again into sixtieths (called respectively *partes minutae* and *partes minutae secundae*, now abbreviated into "minutes" and "seconds") is peculiar and sometimes troublesome but not exactly inconvenient, though a decimal subdivision of the degree would be simpler.

As to time, the fundamental unit is the **day** or period of the earth's rotation (this being the most uniformly moving thing we know). Its subdivisions (into 24 parts, and then into sixtieths, etc.) are curious, but too deep-rooted for anyone to attempt to alter; and fortunately they are the same in all countries.* Legitimate practice in dealing with different denominations can therefore be afforded to children by our large admixture of universally understood measures of **time**; including weeks, months of different kinds, years of different kinds, and centuries. All other weight and measure complications, especially those of a merely insular and boorish character, should forthwith cease to be instilled into children.

Further exercises.

It is worth noticing and remembering that a kilometre $= 10^5$ centimetres.

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* A third subdivision, the sixtieth part of a second, is sometimes known as a "trice."

It is also ten minutes' walk, or very roughly two-thirds of a mile.

A cubic metre is a million cubic centimetres.

A cubic kilometre is a trillion cubic millimetres; meaning by "trillion" a million million million, after the English custom. (But the French use the term "billion" to signify a thousand million; and a million million they accordingly call a trillion; while the above number would by them be designated a quintillion: in any case it is 1 followed by eighteen ciphers).

A cubic centimetre is 1000 cubic millimetres, and is $\frac{1}{1000}$ of a litre.

A gallon of cold water weighs 10 lbs., by definition of a gallon; a pint therefore weighs a pound and a quarter.

A cubic metre of water is a *tonne*, and very approximately, though accidentally, equals an English ton also.

A cubic centimetre of water, at its temperature of maximum density, weighs a gramme exactly, from the definition of a gramme.

The speed of an express train, 60 miles an hour, is only 15 times a walking pace.

The speed of a bullet, say 1800 feet a second, is twenty times that of a train.

The speed of sound is comparable with that of bullets.

The speed of light is a million times the speed of sound in air.

Four miles an hour is 2 yards a second, approximately, or accurately 60 miles an hour is 88 feet a second.

It is an instructive exercise to let a boy find out the sizes and distances of the planets of the solar system, and calculate a numerical model illustrating them on any convenient scale.

I have myself found a local topographical scale the most convenient: one on which the earth was about the size of a football, and the sun the size of some public building a mile or two distant. The other planets distribute themselves about the town and county; some of them extending into more distant counties.

It is instructive to try to place the nearest fixed star in such a scale, and to find that it will not come on to the earth at all.

The price of a railway ticket to the nearest fixed star, at 1d. per hundred miles, can also be calculated; and found to approach or exceed the National debt.

The earth takes a year to go round the sun in a circle of 93 million miles radius: how fast does it go?

(Ans.: About 19 miles a second.)

Light goes 10000 times as fast as this.

How fast would a train have to run on the equator if it were to keep up with the apparent motion of the sun, so that it should continue the same time of day?

(Ans.: About 1000 miles an hour.)

How far from the North Pole could the same thing be accomplished by a man walking 4 miles an hour?

(Ans.: About 30 miles away.)

If a man walked 30 miles South from the North Pole, and then walked 40 miles due West, how far, and in what direction, would he have to go to get back to the Pole?

(Ans.: 30 miles due North.)

What is the density of a rectangular block whose height is 5 inches, length 11 inches, breadth 8 inches, and weight $82\frac{1}{2}$ lbs.?

(Ans.: 3 ounces per cubic inch.)

Directly the elements of mechanics and of heat and of chemistry have been begun, any number of useful and fairly interesting examples can be constructed. They afford practice in arithmetic of the best and most useful kind; quick and ingenious computation being what is wanted, not laborious dwelling upon long artificial sums. Long sums are never done in adult practice: there are always grown-up methods of avoiding them.

It is cruel to subject children to any such disciplinary process, as part of what might be their happy and stimulating education. Before they have been subjected to it they are often eager to have lessons; but experience of the average lesson, as often administered, soon kills off any enthusiasm, and instils the fatal habits of listlessness and inattention which check the sap of intellectual growth for a long time.

If the teacher of arithmetic knows arithmetic and nothing else, he is not fit to teach it. His mind should be alive with concrete and living examples; and it is well to utilise actual measurings, weighings, surveyings, laboratory-experiments, and the like, to furnish other opportunities for arithmetical exercises.

Arithmetical exercise can be obtained unconsciously, as bodily exercise is obtained by playing an outdoor game. The mechanical drill or constitutional-walk form of exercise has its place doubtless, but its place among children is limited. There used to be too much of it, and too little spontaneity of bodily exercise, in girls' schools. Now the spontaneity and freshness is permitted to the body, but too often denied to the mind.

The same kind of reform is called for in both cases. The object of this book is to assist in hastening this vital reform.

CHAPTER VIII.

Simple proportion.

ANY number of sums are of the following character:

If 3 sheep cost £20, what will 100 cost?

Now the so-called "rule of three" method of dealing with sums of this kind, though permissible, is not really a good method, because it leads to nothing beyond, and employs an antiquated system of notation.

The answer is one hundred thirds of twenty pounds $= \frac{100}{3} \times £20 = \frac{2000}{3} = £666{\cdot}\dot{6} = £666\frac{2}{3} = £666.\ 13s.\ 4d.$

If the answer is not obvious, it can be arrived at by the intermediate step of considering one sheep, which will cost the third of £20, namely, £6. 13s. 4d.*

And so a hundred sheep will cost 600 pounds, 1300 shillings, and 400 pence.

The 1300 shillings reduce to 65 pounds, since 100 shillings is five pounds; and the 400 pence make £1. 13s. 4d., since 240 pence is a pound, and so 400 pence is thirty shillings and 40 pence (or 3s. 4d.) over.

This is not an orthodox way of doing the sum, but it is just as good as any other, and it is one that a boy might

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* [It would not come out even so well as this but for the fortunate duodecimal division of the shilling into pence, so that one-third of a pound, viz. 6s. 8d., and two-thirds, viz. 13s. 4d., can be exactly specified without fractions. These amounts are worth remembering as one-third and two-thirds of a sovereign.]

scheme for himself. There would be no need to snub him for it. Everything which is troublesome about such a sum results from the miserable property of the number ten, that it is not divisible by 3.

If we had set the following very similar question:

If 3 sheep cost £24, what would 100 cost?

An infant could answer £800, doing it in its head. But it would clearly do it by the same process, viz. the process of considering the price per single sheep, and that is therefore the natural and simplest method.

To summarise: The childish method is the method by units, and may be written out at length; the adult method is the method by ratio; what place is there for the rule of three? The rule of three with its symbols : :: : is reserved for antiquated school instruction.

Observe, there is no harm in writing a ratio as $2:3$ or $a:b$, and occasionally it may be convenient to do so, though $2 \div 3$, or $a \div b$ is precisely the same thing, and usually the form $\frac{2}{3}$ or $\frac{a}{b}$, or a/b, is in every way better. So the symbol :: is needless, because replaced by $=$. The fact is that : connotes the theoretical idea of ratio, while $\div$ indicates the practical operation of division, which is the actual means of working a ratio out. The vulgar-fraction form may be used instead of either of these signs and is usually best. The division may or may not be actually performed, as we please.

I feel inclined to illustrate good and bad methods at this stage a little further, by taking a few more very simple examples. For instance:

If twenty dogs pulling equally at a sledge exert a horizontal force of 1 cwt., what force do any three of them exert?

Adult method :

$$\frac{3}{20}\text{ths of 1 cwt.} = \frac{3 \times 112 \text{ lbs.}}{20} = 16{\cdot}8 \text{ lbs.}$$

Good childish method :

20 dogs pull 112 lbs.
10 dogs pull 56 ,,
1 dog pulls 5·6 ,,
3 dogs pull $3 \times 5{\cdot}6 = 16{\cdot}8$ lbs.

If it be asked why not stop at $\frac{3}{20}$ths of a cwt. and give the answer as ·15 cwt., I reply, no reason against it at all; but children should be accustomed to realise forces and other things, in actual homely units that they can feel and appreciate; and a cwt. is too big for them.

Mechanical method :

20 : 3 :: 112 : the answer.

RULE. Multiply the means and divide by one extreme and you get the other extreme.

∴ the answer is, etc.

British Method :

There is indeed a barbarous way of complicating the sum, which is typical of much that goes on in these islands at inferior schools :

	lbs.	oz.	drachms
20	336 .	0 .	0
	16 .	12 .	$12\frac{4}{5}$

which is done thus :

Twenty into 336 goes 16 and 16 over, that is 16 lbs. over, which equals 256 ounces. Twenty into this goes 12 times and 16 over, that is 16 ounces or 256 drachms ; into which twenty again goes 12 times and $\frac{16}{20}$ths over, which last equals $\frac{4}{5}$ths, that is $\frac{4}{5}$ths of a drachm.

So the answer is 16 lbs. 12 oz. $12\frac{4}{5}$ drachms.

On this one has to remark that since the unfortunate $\frac{4}{5}$ has to appear (as it happens) sooner or later, why should it not appear at first? Why is $\frac{4}{5}$ths of a drachm easier to understand than $\frac{4}{5}$ths of a pound? The fact is that it is not easier to understand, and by children is not understood : the "4 over" which remains at the end is a continual puzzle to them.

They have been so accustomed to getting rid of fractions by reducing to a lower denomination, that at the end, when lower denominations unaccountably fail them, they are nonplussed. Quite rightly so; the fault is not with the children.

Whenever an attentive child finds a persistent difficulty, teachers should be sure that there is something wrong with their mode of presenting it, probably with their own comprehension of it. Nothing is difficult when properly put. The whole art of teaching should be so to lead on that everything arrives naturally and easily and happily, like fruit and flowers out of seeds.

Another British method. Usually however the sum is not recorded so briefly as this, but is written out in what is known as the long-division plan; and it is perhaps the safest mode of getting the right answer if the answer is required to be thus barbarously specified, for it certainly shirks nothing. This is the way of it:

To divide 336 lbs. av. into 20 equal parts

lbs. oz. dr.

2,0) 33,6 (16 . 12 . $12\frac{4}{5}$

20

136

120

16

16*

96

16

256 oz.

240

16

16*

96

16

256 dr.

240

16 remainder, and $\frac{16}{20} = \frac{4}{5}$ dr.

* If any mathematician glances through this book, as I hope he may, he will require at these stages to be reminded if British, to be informed

This may look like a parody, but it is soberly the way in which innumerable children have been taught in the past to do such a sum. And the fact that they have been so taught can easily be tested by setting it to people who were children a few years ago.

Another method. If the **factor** plan of division is adopted there is great danger of confusion and error about the carrying figure. For instance, in dividing 336 lbs. into 20 equal parts, a child as sometimes now taught will proceed thus:

```
  2 | 336 lbs.
 10 | 16,8
      16 and 8 over.
```

8 what over? They are apt to take it as 8 lbs. over, and so interpret it as 128 ounces, and proceed to divide these again by 20 by the same process

```
  2 | 128
 10 |  64
       6 and 4 over
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apt to be called 4 ounces over, which are interpreted as 64 drachms, and so on.

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if Foreign, that in these islands a drachm is defined to be the sixteenth of an ounce, and that an ounce avoirdupois is one sixteenth of an avoirdupois pound; moreover that a drachm is the lowest recognised denomination of avoirdupois weight: after that fractions are permitted. Pennyweights and grains belong to a system of measures to which the name of "Troy" is (for some to me unknown reason, perhaps from Troyes in France) prefixed. There is a "Troy pound" and a "Troy ounce," for "metallurgical" use, but they differ from their "grocery" cousins which are explicitly asserted "to have some weight." Then between grains and Troy ounces there are other denominations used by "apothecaries," called scruples and drams. This dram is not the same as the grocery drachm. There appears however to be only one kind of "grain," and 7000 of these make 1 lb. avoirdupois, while 5760 of them make 1 lb. Troy.

This is quite wrong. The 8 over in the first little sum was really 8 double-pounds, and so the second little sum is all wrong. If it had been right, the 4 over could not have been 4 ounces, but 4 double-ounces; but what needless trouble and risk of error is introduced by having to perceive this!

Again let many children be asked to divide £336 by 25, they will few of them have been taught to proceed thus:

$$\frac{336}{25} = 3{\cdot}36 \times 4 = £13{\cdot}44$$
$$= £13.\ 8{\cdot}8s.$$
$$= £13.\ 8s.\ 9{\cdot}6d. \text{ or about } 9\tfrac{1}{2}d.$$

but they will proceed, either by long division on much the same lines as in the last example, which is long to write, or else by short division, dividing by 5 twice over, which is not too long to write,

	£	s.	d.
5	336	0	0
5	67	4	0
	13	8	$9\frac{3}{5}$

short to write, but rather hard to do. Such trivial sums should not call for so much brain power as is involved in various and complicated carryings.

Money sums however are the best examples of the kind. If it was 336 **tons** that had to be divided into 25 equal parts, grown people would be satisfied to say that each part must be 13·44 tons; but at some schools it would have to be done thus,—if not by a still longer process equally liable to accidental error:

	tons.	cwt.	qrs.	lbs.	ozs.	dr.	
5	336	0	0	0	0	0	
5	67	4	0	0	0	0	
	13	8	3	5	9	$9\frac{3}{5}$	Ans.

Breakdown of simple proportion or "rule of three."

Simple proportion, or the rule of three, is by some teachers regarded as a kind of fetish; moreover its extreme simplicity makes it a rather favourite rule with children and they will naturally do many exercises in it. Not always, it is to be hoped, by the same mechanical method.

But there is all the more necessity for bringing home to them the fact (strange if it is unknown to any teacher), that it does not always work. For instance:

A stone dropped down an empty well 16 feet deep reaches the bottom in one second. How deep will a well be if a stone takes two seconds to reach the bottom?

The answer expected is of course 32 feet; but it is not right. The correct answer is 64 feet.

If a stone drops 16 feet in one second, how far will it drop in $\frac{1}{4}$ second? (Ans.: 12 inches.)

Again, if a stone dropped over a cliff descends 64 feet in 2 seconds, how far will it drop in the next second? (Ans.: 80 feet.)

A steamer is propelled at the rate of 8 knots by its engines exerting themselves at the rate of 1000 horse power. What power would drive it at 12 knots?

Probably no one would expect the answer 1500 to this; for on that principle 10000 horse power would propel it at 80 knots.

An initial velocity of 1600 feet a second will carry a rifle bullet 3 miles. What velocity would carry it 6 miles?

An ounce weight drops 4 feet in half a second. How far will a pound weight drop in the same time?

(Ans.: By experiment, 4 feet likewise. A most important fact, discovered by Galileo, and illustrated from the tower of Pisa.)

Let it not be dogmatised on, but illustrated by dropping things together; and if it appears puzzling, so much the better. Ignoring or eliminating the resistance of the air everything falls at the same pace. The air has very slight influence on the drop of smooth spheres through a moderate height. Cotton wool and feathers and bits of paper will drop more slowly, but the reason is obvious: a bullet will drop more slowly in treacle than in air. That is because the air resistance is small: it is not zero, and if a bullet and a pea were dropped from too great a height, air friction would begin perceptibly to retard the lighter body. So it is that big rain-drops fall quicker than little ones; and these small drops quicker than mist and cloud globules. So also does heavy fine powder, even gold powder, fall slowly in water, not because it is buoyed up, but because it is resisted. Remove the air, and in a vacuum a coin and a feather will fall at the same rate. The statement does not explain the fact. The full explanation of the fact is not even yet known. But a very great deal more is known about the whole subject than is or can be here expressed. That is characteristic of elementary books throughout, and the object of the learners should be to get through all this easy stuff, and get on into more exciting matters beyond: matters which the majority of the human race never have the least knowledge of, because their early education has been neglected.

A balloon 18 feet in diameter can carry a load equal to one man. What load can a similar balloon carry which is 36 feet in diameter. (Simplest rough answer, 8 men.)

A rope stretches half an inch when loaded with an extra hundredweight.

How much would it stretch if loaded with an extra ton?

A half crown is ten times the value of a threepenny bit. How many threepenny bits can lie flat on a half-crown without overlapping the edge? (Ans.: By experiment, one.)

A boy slides 20 yards with an initial run of 10 feet. What initial run would enable him to slide half a mile?

If 2 peacocks can waken one man, how many can waken six?

If a diamond is worth ten thousand pounds, what would 950 similar diamonds be worth?

If a camel can stand a load of 5 cwt for 6 hours, for how long could he stand a load of ten tons?

These things cannot be done by simple proportion. They require something more to be known before they can be done at all; and accordingly it would appear as if generations of teachers had discreetly shied at them all, indiscriminately, and had excluded them from arithmetical consideration altogether. It is just as if in geometry, finding straight lines simpler than curves, they had agreed to found all their examples upon straight lines.

CHAPTER IX.

Simplification of fractions.

VULGAR fractions are much harder to deal with than decimals; but as sometimes several have to be added together it is desirable to know how to do it. Besides, the exercise so afforded is of a right and wholesome kind.

Consider the following addition: $\frac{1}{2}+\frac{1}{4}$. Small children can see (by experiment on an apple) that the result is $\frac{3}{4}$, and they can also be taught to regard it as $\frac{2}{4}+\frac{1}{4}=\frac{3}{4}$, which should be read in words—two quarters added to one quarter make three quarters.

Thus, it can be realised that when the denominators are all the same, addition of fractions becomes simple addition of the numerators.

For just as 5 oranges + 6 oranges = 11 oranges, so

$$\frac{5}{17}+\frac{6}{17}=\frac{11}{17},$$

reading "seventeenths" instead of "oranges."

When denominations differ, therefore, the first thing to do is to make them the same.

Thus, for instance, 3 apples + 4 oranges, is an addition which can only be performed by finding some denomination which includes both, say "pieces of fruit."

So also 7 horses + 3 pigs = 10 quadrupeds. 5 copies of *Robinson Crusoe* + 3 copies of *Ivanhoe* = 8 prize-books, perhaps.

Reduction to the same denomination cannot always be done, when denominations are anything whatever, except by using the vague term "objects" or "things"; but with numerical denominators it can always be done, and the method of doing it has to be learnt. $\frac{3}{7}+\frac{5}{14} = \frac{11}{14}$, and such like, are easy examples. $\frac{1}{4}+\frac{1}{12} = \frac{1}{3}$ is a slightly harder one.

It is done by saying $\frac{3}{12}+\frac{1}{12} = \frac{4}{12} = \frac{1}{3}$.

So also $\frac{1}{2}+\frac{1}{6} = \frac{2}{3}$, being equal to $\frac{4}{6}$.

A harder example is $\frac{7}{2}+\frac{5}{3}$, which can be written

$$\tfrac{21}{6}+\tfrac{10}{6} = \tfrac{31}{6} = 5\tfrac{1}{6}.$$

In the decimal notation this would appear thus:

$$3{\cdot}5+1{\cdot}666\ldots = 5{\cdot}1666\ldots.$$

A still harder example can be worked out thus:

$$\tfrac{9}{8}+\tfrac{5}{7} = \tfrac{63}{56}+\tfrac{40}{56} = \tfrac{103}{56} = 1\tfrac{47}{56},$$

though the final step is one that need not always be made.

Now it is evident, or at least it will gradually be found true, that in a mechanical process of this kind there is always some simple rule by which the result can be obtained without thought. What is that rule? If the child can find it out for himself, by experimenting on lots of pairs of fractions, so much the better. A week is none too much to give him to try, for if he finds it out himself he will never forget it.

The rule is: cross-multiply for the numerators, and multiply the denominators.

$$\frac{1}{2}+\frac{1}{6}=\frac{6+2}{12}=\frac{8}{12}=\frac{2}{3}.$$

$$\frac{1}{a}+\frac{1}{b}=\frac{b+a}{ab}.$$

$$\frac{3}{7}+\frac{4}{9}=\frac{27+28}{63}=\frac{55}{63}.$$

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd},$$

but it would be a pity to spoil this by premature telling.

The fact that **the sum of two reciprocals is the sum of the numbers divided by their product**, is worth illustrating fully and remembering: remembering, that is, by growing thoroughly accustomed to it, not exactly learning by heart. There is hardly any need to learn easy things like that by heart: nevertheless it is a very permissible operation, whenever the fact to be learnt is really worth knowing.

$$\frac{1}{3}+\frac{1}{4}=\frac{7}{12}, \text{ that is the } \frac{\text{sum}}{\text{product}}.$$

$$\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}.$$

$$\frac{1}{23}+\frac{1}{5}=\frac{28}{115}.$$

$$\frac{1}{2}+\frac{1}{49}=\frac{51}{98}\triangleq\frac{52}{100}=\cdot 52.$$

the symbol $\triangleq$ meaning "approximately equals."

[The approximation is seen to be true because adding 1 to 50 makes the same proportional difference as adding 2 to 100. If this is too hard, it can be postponed. It is unimportant, but represents a kind of thing which it is often handy to do in practice.]

But this rule of cross-multiplication hardly serves for the addition of three or more fractions, at least not without modification. Take an example,

$$\frac{1}{6}+\frac{2}{3}+\frac{7}{2}=\frac{1}{6}+\frac{4}{6}+\frac{21}{6}=\frac{26}{6}=4\tfrac{1}{3}.$$

Take another, $$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{4+2+1}{8}=\frac{7}{8},$$

where the three fractions $\frac{4}{8}$, $\frac{2}{8}$, and $\frac{1}{8}$, all having the same denominator, are written all together, with the addition of the numerators indicated, and subsequently performed.

One more, $$\frac{1}{3}+\frac{1}{2}+\frac{1}{9}=\frac{3+4\frac{1}{2}+1}{9}=\frac{8\frac{1}{2}}{9}=\frac{17}{18}.$$

This might hardly be considered a legitimate procedure, but there is nothing the matter with it. You might, instead, proceed thus:

$$\frac{1}{3}+\frac{1}{2}+\frac{1}{9}=\frac{18}{54}+\frac{27}{54}+\frac{6}{54}=\frac{51}{54}=\frac{17}{18},$$

and that is equally a correct method.

But neither of these plans is quite the grown-up plan. Let a better plan be found; but first let the above plans be formulated and expressed. Is it not plain that the numerator of each particular fraction is found by multiplying two of the denominators together, while the common denominator of all the fractions is found by multiplying all the denominators together? Apply this rule:

$$\frac{1}{6}+\frac{1}{5}+\frac{1}{4}=\frac{20+24+30}{120}=\frac{74}{120}=\frac{37}{60}.$$

For instance, a sixth of an hour + a fifth of an hour + a quarter of an hour = 37 minutes, a minute being the sixtieth of an hour. Now a sixth of an hour is ten minutes, a fifth is 12 minutes, and a quarter of an hour is 15 minutes: consequently the neatest way of doing the sum would be

$$\frac{1}{6}+\frac{1}{5}+\frac{1}{4}=\frac{10+12+15}{60}=\frac{37}{60}.$$

Another example, $$\frac{1}{12}+\frac{1}{60}+\frac{1}{3}=\frac{180+36+720}{720\times 3},$$

but here every term in numerator and denominator can be divided by 3 and by 12, so that the above may be written

$$\frac{1}{12}+\frac{1}{60}+\frac{1}{3}=\frac{5+1+20}{60}=\frac{26}{60}=\frac{13}{30}=\cdot 04\dot{3}.$$

And it would have been neater to write it so at first—neater but not essential, and sometimes not even the most rapid plan.

To illustrate the above example:

$\frac{1}{12}$th of a day is 2 hours.

$\frac{1}{60}$th of a day is 24 minutes.

$\frac{1}{3}$rd of a day is 8 hours.

Consequently the sum of these fractions of a day is 10 hours and 24 minutes,

which is $10\frac{24}{60}$ of an hour [$= 10\frac{4}{10} = 10{\cdot}4$ hours] or $\frac{10}{24}+\frac{1}{60}$ of a day,

which again may be written $\frac{25}{60}+\frac{1}{60} = \frac{26}{60} = \frac{13}{30}$ths $= {\cdot}04\dot{3}$, as before.

The form of the general rule, then, is given by

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{bc+ca+ab}{abc};$$

but in practice it is possible to abbreviate this in some cases, when one of the denominators contains the others as factors, or when some simple relation of the kind exists between them.

This is what was made use of in the early simple cases, such as $\frac{1}{12}+\frac{5}{24}$; we did not proceed to write $\frac{24+60}{288}$ and then simplify it, but we wrote at once $\frac{2}{24}+\frac{5}{24} = \frac{7}{24}$; that is to say we perceived that 24 would do for the new denominator, and we adjusted the numerators accordingly.

Perhaps we had better display this algebraically. Let each denominator contain a common factor, say n, so that the reciprocals to be added are $\frac{1}{na}+\frac{1}{nb}+\frac{1}{nc}$, then if we applied the mere general rule we should write $\frac{n^2bc+n^2ca+n^2ab}{n^3abc}$, but the repetition of the powers of n is manifestly needless, since they cancel out; and it is much neater to write for the new denominator an expression which contains the common factor n only once, thus: $\frac{bc+ca+ab}{nabc}$.

The denominator so obtained is called the **least common multiple** of the three denominators; and it is frequently, in examination papers, denoted by the letters L.C.M. It is not an important idea at all. Sums can be done quite well without it, but its introduction affords some scope for neatness and ingenuity. Easy processes can be given for finding it, but they are hardly worth giving, as in real practice they are seldom used: they are of most educational service if employed as an exercise for the student's invention. They will be dealt with sufficiently in the next chapter.

Now take a numerical example:

Add together $\frac{1}{2}+\frac{1}{4}+\frac{5}{8}+\frac{3}{16}+\frac{7}{32}$.

Here 32 is evidently the L.C.M. of the denominators, since it contains all the others as factors. So that will serve as the simplest common or combined denominator. The first numerator accordingly will be 16, the second 8, the third 4 but taken 5 times and therefore 20, the next 2 taken 3 times, and the last 1 taken 7 times.

Consequently the sum is written as follows:

$$\frac{1}{2}+\frac{1}{4}+\frac{5}{8}+\frac{3}{16}+\frac{7}{32}=\frac{16+8+20+6+7}{32}=\frac{57}{32}.$$

Take another example of addition:

$$\frac{1}{7}+\frac{1}{56}+\frac{1}{9}+\frac{1}{63}=\frac{72+9+56+8}{504}=\frac{145}{504}.$$

Here 7 is plainly a factor of both the larger denominators, and 8 and 9 are the other factors, so the least common denominator will only contain 7 ard 9 once, and will equal $7\times 8\times 9=504$, and this being the smallest common multiple possible, no further simplification can be effected; beyond of course expressing the result as a decimal if we so choose. To express it as a decimal we must effect the division indicated; the result happens to equal ·2877 almost exactly.

It is worth noticing that the series of powers of $\frac{1}{2}$, viz.:

$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}+\tfrac{1}{64}+\dots$$

add up very nearly to 1; and the more nearly the more terms of the series are taken.

It can be shown, not by trial indeed, but by simple reasoning, that if an infinite sequence of this series are added together the result is exactly 1. Thus the first term constitutes half of the whole quantity, say a loaf, the second term added to it gives us three quarters, the third term gives us $\frac{1}{8}$th more, and we only need another eighth to get the whole. The next term gives us half of the deficiency, and now we need the other sixteenth to make the whole. We do not get it however: we get half of it in the next term, and thus still fall short, but this time only by $\frac{1}{32}$; and so at the end of the above series, as far as written, our deficiency is $\frac{1}{64}$th. **Each term therefore itself indicates the outstanding deficiency,** and as the terms get rapidly smaller and smaller, so does the deficiency below 1 get rapidly diminished till it is imperceptible. (Compare p. 325.)

It is convenient to plot these fractions as lengths (setting them up at equal distances along a horizontal line), say half a foot, then a quarter, then an eighth, and so on. Then joining their tops we get a curve which has the remarkable property of always approaching a straight line, but never actually meeting or coinciding with it, or at least not meeting it till infinity; when at length it has become quite straight.

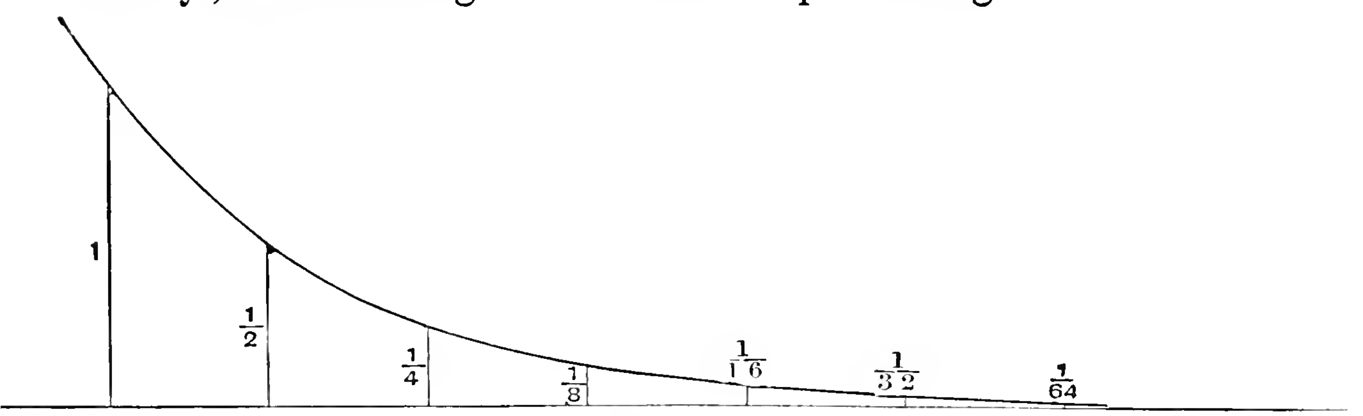


FIG. 9.

There are many curves with such a property, but fig. 9 may be the first a child has met. He can of course continue the curve in the other direction—the direction of whole numbers, or powers of two, and observe how rapidly it tilts upwards; but there is no straight line in this direction to which it tends to approach; this end proceeds to infinity both upwards and sideways, not only upwards, though it proceeds far more rapidly in the vertical direction than in the horizontal; and this end of it never becomes straight.

CHAPTER X.

Greatest Common Measure and Least Common Multiple.

ANOTHER name of slight importance, which is usually paired off with Least Common Multiple (page 99), is Greatest Common Measure or Highest Common Factor: often denoted by G.C.M. or by H.C.F.

The two numbers 24 and 16 have several factors common to both of them, for instance 8; and this as it happens is the greatest common factor, the others which they possess in common being 4 and 2.

The numbers 20 and 35 have 5 as the largest factor common to both of them. The numbers 72 and 84 have 12; while 72 and 96 have 24 as their G.C.M.

The numbers 23 and 38 have no factor, above unity, common to both. In fact 23 has no factor at all.

The word "common" so used does not mean "ordinary," as children sometimes think, nor does it mean vulgar, but it has the signification which it possesses in "common friend," or in vulgar phrase "mutual friend," or when people are said to own property "in common."

To find common factors of two numbers, one way is to arrange all the factors of each in two rows one under the other and see how many correspond. Inspection will then readily show which pair is the biggest.

Suppose the two numbers given were 40 and 60; the following are the factors of 60,

2, 3, 4, 5, 6, 10, 12, 15, 20, 30,

and the following are the factors of 40,

2, 4, 5, 8, 10, 20.

Of these, the 2, 4, 5, 10, 20 are **common** to both, and 20 is the largest of them.

In old-fashioned language, factors were called "measures," and the largest common factor was called the "greatest common measure," and abbreviated into G.C.M.

What is the use of it? Very little; but the meaning is perfectly simple and should be understood. It can be utilised for finding the Least Common Multiple of a set of numbers, that is to say the smallest number which contains them all as factors; for the G.C.M. represents what may be struck out, once at least and sometimes more than once, from the product of a set of numbers, in order to leave behind the smallest number which they are able to divide without a remainder.

Thus take the numbers 40 and 60, their product is 2400, and of course they will both divide that; but their G.C.M., 20, may be cancelled out of it, leaving 120; and both 40 and 60 will divide that too. It is the least number which they can both divide exactly, *i.e.* it is the least number of which they are both factors, it is in fact their **least common multiple.**

Example.—Of the numbers 12, 20, 24 what is the G.C.M. and L.C.M.? Of these, 12 need not be attended to in finding the largest common factor, because it is itself a factor of 24.

Of the numbers 20 and 24, 4 is a common factor; so divide all by that, and we get left with 3, 5, 6.

No factor will divide all these, so 4 was the G.C.M. of the original numbers.

Their least common multiple is not $3 \times 5 \times 6 = 90$, because that would have omitted the factor 4 which they possess in common. The common factor need not be repeated more than once, (for if it is, though you get a common multiple, you do not get the least common multiple), but it must not be omitted altogether, or you will not get a common multiple at all.

The L.C.M. accordingly is $3 \times 5 \times 6 \times 4 = 360$, and of that it will be found that the given numbers 12, 20, 24, are factors. The product of those numbers is 5760, and out of that the G.C.M. 4 can be struck twice before arriving at the L.C.M.

Anyone therefore can invent a rule for finding the L.C.M. of a set of numbers; it is, find their G.C.M. and divide or cancel it out of all the numbers but one, then multiply the quotients together.

But a rule for finding the G.C.M. is by no means so easy to invent: it is an ingenious process, and the whole subject is essentially a little bit of rudimentary pure mathematics; it has no practical importance or application except when dealing with the properties of numbers.

The proof of the rule is an interesting and easy exercise in the application of reason and commonsense to arithmetic, but perhaps it is better deferred.

Rule for finding G.C.M.

The rule depends on the demonstration that **any factor of two numbers is likewise necessarily a factor of the remainder left when one is divided by the other.**

Thus consider the two numbers 40 and 24. Divide one by the other, we get 1 and 16 over. The above sentence in black type assumes or asserts that every factor of 40 and 24 must also be a factor of 16. In this case, as a matter of fact,

$$40 = 24 + 16$$

and it is manifest that a number which divides 24 and does not divide 16 cannot divide 40.

Well that is the whole idea.

If we were told to find the G.C.M. of 40 and 24, we could by this means reduce the problem to finding the G.C.M. of 24 and 16. And then, repeating the division process, we should observe that

$$24 = 16 + 8,$$

so that the problem becomes reduced still further into finding the G.C.M. of 16 and 8. There is no question but that this is 8;—as indeed we might have guessed at first if our object had been attainment of a result, instead of explication of a process—and the way to clinch that is to perform the division again and to find that there is now no remainder at all.

The matter can be stated algebraically, but beginners can skip the algebra and come to the "illustration" which follows.

Algebraic proof of the process for finding G.C.M.

To find a common factor of two numbers P and Q, of which P is the bigger,

let x be one common factor,

then $\frac{P}{x}$ and $\frac{Q}{x}$ will be the complementary factors.

An extreme case is when P is divisible by Q without a remainder, in that case $x = Q$. Suppose however that when P is divided by Q the remainder is R,

$$\begin{array}{l} Q\,)\,P\,(\,n \\ \quad\; \underline{nQ} \\ \quad\;\; R \end{array}$$

so that $P = nQ + R$; then if R is a factor of Q it must be one of P also (because P equals a multiple of Q plus R), so try if R is a factor of Q.

If it is, it is the common factor required; but if not, work out a division again, and let the remainder be S,

$$\begin{array}{l} R\,)\,Q\,(\,m \\ \quad\; \underline{mR} \\ \quad\;\; S \end{array}$$

so that $Q = mR + S$.

Then if S is a factor of R it must be one of Q too, and so also of P, and in that case S will be the common factor required.

But if not, we must repeat the process and see what the remainder is when R is divided by S. Call it T,

$$\begin{array}{r} S)\,R\,(l \\ \underline{lS} \\ T \end{array}$$

so that $R = lS + T$.

Now once more if T is a factor of S it is necessarily a factor of R, and therefore of Q, and therefore also of P, and so T is the common factor required.

If not, the process must go on until there is no further remainder; and then the last remainder (or divisor) is a common factor of the two original numbers P and Q. Let us assume that T divides S without a remainder, then T is the common factor of all the numbers P, Q, R, S, T.

It is likewise the largest common factor which exists. Why? because it has to be a factor not only of P and Q but also of R, of S, and of T; and certainly T is the largest factor of T, therefore it is likewise the largest common factor of the others.

Statement in another form.

The whole process can be written thus:

To find the G.C.M. of P and Q, work successive division sums thus:

$$\frac{P}{Q} = n + \frac{R}{Q},$$

$$\frac{Q}{R} = m + \frac{S}{R},$$

$$\frac{R}{S} = l + \frac{T}{S},$$

or $$\frac{P}{Q} = n + \frac{1}{Q/R} = n + \cfrac{1}{m + \cfrac{1}{R/S}} = n + \cfrac{1}{m + \cfrac{1}{l + \cfrac{1}{S/T}}},$$

the process terminating only when S/T is an integer.

The T is a factor of all the numbers P, Q, R, S, T; and since it *must* satisfy this condition if it is to be a factor of P and Q at all, it is necessarily the greatest common factor of P and Q, and indeed of the others too.

Or the whole process may be written (as usually performed) in one sum thus:

$$\begin{array}{l} Q)\,P\,(n \\ \quad \underline{nQ} \\ \quad R)\,Q\,(m \\ \qquad \underline{mR} \\ \qquad S)\,R\,(l \\ \qquad\quad \underline{lS} \\ \qquad\quad T)\,S\,(k \\ \qquad\qquad \underline{kT} \\ \qquad\qquad \ldots \end{array}$$

Then the last remainder (or divisor) T is the G.C.M. of P and Q.

Illustration (modified from Kirkman and Field).

Let the two numbers be 492 and 228. Go through a process of successive divisions.

$$\begin{array}{l} 228)\,492\,(2 \\ \qquad \underline{456} \\ \qquad\ 36)\,228\,(6 \\ \qquad\qquad \underline{216} \\ \qquad\qquad\ 12)\,36\,(3 \\ \qquad\qquad\qquad \underline{36} \\ \qquad\qquad\qquad \ldots \end{array}$$

Hence 12 is the G.C.M. of the two original numbers, and it likewise is a factor of the intermediate divisor, viz. 36.

The argument runs as follows:

The common factor of 492 and 228 must also be a factor of the remainder when 492 is divided by 228, for in fact

$$492 = (2 \times 228) + 36,$$

so that anything which divides 228 and fails to divide 36 cannot possibly divide 492.

Hence the problem reduces itself to finding the common factor of 228 and 36.

But now $$228 = (6 \times 36) + 12,$$
hence the factor required must likewise divide 12, as well as 36. The numbers 2, 3, 4, 6, 12 all satisfy that condition, and hence all these are factors of both the original numbers, but of them 12 is the biggest.

Therefore 12 is the G.C.M. of the two given numbers 492 and 228. (Verify this by actual division. The quotients are 41 and 19, and neither of these have any factors at all.)

CHAPTER XI.

Easy mode of treating problems which require a little thought.

MANY of the problems set for purposes of arithmetic are best done in the first instance by rudimentary algebra, that is by the introduction of a symbol for the unknown quantity, so that it can be tangibly dealt with. This introduction and manipulation of a symbol for an unknown quantity need not be discouraged, even from the first. It confers both power and clearness. Many arithmetical sums are needlessly hard because x is forbidden. There is a certain amount of sense in the artificial restriction, but in complicated sums and in physics the symbolic treatment of unknown quantities is essential, and the sooner children are accustomed to it the better.

The introduction of a symbol for an unknown quantity is a device to enable a sum to be clearly and formally stated. After the sum has been solved by this aid, it is well to try and express it so that it can be grasped and understood without such assistance. The fear of those who object to x in arithmetic is that this final step may be omitted. The grasp is clearer when an auxiliary symbol can be dispensed with; but that is not possible always at first. The x is to be thought of as a kind of crutch: but sometimes it is like a leaping-pole and enables heights to be surmounted which without it would be impossible.

Example.—How soon after twelve o'clock will the hour and minute hand of a clock again be superposed?

It is plain that it is soon after 1 o'clock, and that it is an amount which has been traversed by the hour hand while the minute hand, travelling twelve times as quickly, has gone that same distance and 5 minutes more; but it is not easy to think out the required fraction in one's head, though exceptional children can do it.

But let it be postulated as n minutes after 1; the hour hand travels, starting from mark I, a distance n, while the minute hand, starting from mark XII five minutes further back, has to travel $5+n$ in order to catch it up; so the relative speeds of the two hands are as $(n+5):n$, and are also as $12:1$; wherefore

$$\frac{n+5}{n} = \frac{12}{1},$$

or
$$12n = n+5,$$

or
$$11n = 5,$$

or
$$n = \frac{5}{11},$$

and so the time required (or the answer) is five minutes and five elevenths of a minute (*i.e.* $\frac{1}{11}$ hour) past one o'clock.

Take another question.—Start with a clock face indicating 9 o'clock, and ask when the hands will for the first time be superposed.

The slow-moving hand has forty-five minutes' start; so, however many minutes it goes, the quick one has to go 45 minutes more, at twelve times the pace. Wherefore $x+45 = 12x$, or the meeting point is $\frac{45}{11} = 4\frac{1}{11}$ minutes after the mark IX; or $\frac{12}{11}$ths of 45 minutes, *i.e.* $\frac{9}{11}$ths of an hour, since 9 o'clock. The start in this case is nine times as great as was allowed after one o'clock, in the previous question, and accordingly the distance before overtaking occurs is likewise nine times as great: in accordance with common sense.

The constant occurrence of 11 in such sums shows that 11 must have a decipherable meaning: it means the excess pace, or relative velocity, of the quick hand over the slow. And when this has been perceived, the easiest way to do such sums in the head is self-suggested, viz. to treat it as a case of relative velocities, with the hour hand stationary, and simply ask how soon the minute hand will move to where the hour hand *was*, if it (the minute hand) went at $\frac{11}{12}$ths of its real speed.

The interval between *successive* overlaps is therefore always $\frac{12}{11}$ths of an hour, or $65\frac{5}{11}$ minutes.

Exercise.—The hands make a straight line at 6 o'clock, when will they be at right angles? Ans.: One has to gain relatively 15 minutes on the other, and since its relative speed is $\frac{11}{12}$ths of an hour per hour, the time required is $15 \times \frac{12}{11}$ minutes, that is to say $1\frac{4}{11}$ minutes more than a quarter of an hour.

Pains should always be taken to express an answer completely and intelligibly. If any joy is taken in work, it should be decorated and embroidered, so to speak, not left with a minimum of bare necessity.

Moreover, never let it be taught (as Todhunter taught) that the x or other symbol so employed is always necessarily only a pure number. When we say "let x be the velocity of the train," or "the weight of the balloon," etc., we should mean that x is to stand for the **actual** velocity, the **actual** weight: however they be numerically specified. (Appendix II.)

Some teachers of importance will demur to this. I assert with absolute conviction that it is the right plan, and will justify it hereafter. But it is a matter for adults to consider, and is only incidentally mentioned here.

The dislike felt by teachers of arithmetic to the introduction of x prematurely, is because there is a tendency thereafter to do arithmetical problems so easily that their features are not grasped, and so some useful perceptions are missed. If this were a **necessary** consequence it would be a valid argument against the introduction of an algebraic symbol, but it is not a necessary consequence.

For instance, in examples about the supply of a cistern by pipes, or the work of men per day, it is admittedly desirable to realise that we are here often dealing with the **reciprocals** of the specified quantities; and this may be masked by the use of algebra, possibly, but it need not. I suggest that algebra is the right way of discovering the fact, but that after its discovery the fact itself may be properly dwelt on, and thereafter directly applied. There is indeed too much tendency to hurry away from an example when its mere "answer" has been obtained, without staying to extract its nutriment and learn all that it can teach: sometimes without even trying whether the answer found will really fit or satisfy the data in question. That is altogether bad. The full discussion of a sum, in all its bearings, after the answer is known, is often the most interesting and instructive part of the process.

Children should always be encouraged to do this, and to invent fresh ways of putting things, or detect or devise a generalisation of their own for any suitable special case. Here is afforded a first scope for easy kinds of originality of a valuable kind.

Girls especially would find the benefit of being encouraged to seek the general under the mask of the special. It seems to fail to come to them naturally.

Illustrative Examples, showing the advantage of introducing symbols for unknown quantities.

Three pipes supply a cistern which can hold 144 gallons.

One supplies a gallon a minute, another 2 gallons, and the third 3 gallons per minute. How soon will the cistern be full?

Let t be the number of minutes before the cistern is full after the pipes are all turned on simultaneously; then in t minutes the first pipe will have supplied t gallons, the second $2t$ gallons, and so on,

hence $$t + 2t + 3t = 144.$$

So $$t = 24.$$

This is easy enough, but I think even this is made easier by the introduction of a symbol for the unknown quantity.

Take however the following variation of the same problem:

A cistern is to be filled by three pipes labelled A, B, and C;

Pipe A alone would fill the cistern in 2 hours 24 minutes.

Pipe B alone in 1 hour 12 minutes.

Pipe C alone in 48 minutes.

How soon would they all three fill it?

This form of statement evidently makes the problem harder, and it is clearly desirable to simplify it by ascertaining the rate of supply of each pipe. This can be done at once if we say, let n be the number of gallons corresponding to the contents of the cistern, for then the data give us that

Pipe A supplies at the rate of n gallons in 144 minutes

or $\frac{n}{144}$ gallons per minute,

B supplies at the rate $\frac{n}{72}$ gallons per minute,

and C supplies at the rate $\frac{n}{48}$ gallons per minute.

So the set of pipes together supply, at the combined rate,

$$\frac{n}{144} + \frac{n}{72} + \frac{n}{48} = \frac{n}{t},$$

that is to say, n gallons in the unknown time t, which time is the thing to be found.

We now see that the contents of the cistern is immaterial, when the data are thus specified, for n cancels out of the equation, and leaves us with the relation

$$\frac{1}{t} = \frac{1}{48} + \frac{1}{72} + \frac{1}{144}.$$

We have thus discovered the mode of dealing with problems of this kind, viz. to take the *reciprocals* of the times given. In other words, to say that the rate of supply is inversely as the time taken, or that it is proportional to the reciprocal of that time; and hence, writing the combined rate as the sum of the rates, we get the equation *directly* as last written.

Now it is true that a mathematician would have seen this at once, and written the equation as above without appearing to think about it; but a child cannot be expected to think out such a relation, at least not for a long time, unless he is encouraged to consider, either tacitly or explicitly, the contents of the cistern; when it at once becomes, not exactly easy but, possible.

The above equation may be called "the solution" of the problem, so far as it involves reasoning or thought; the subsequent arithmetical working necessary to obtain a numerical result is comparatively mechanical, but it should not be omitted.

$$\frac{1}{48} + \frac{1}{72} + \frac{1}{144} = \frac{3+2+1}{144} = \frac{6}{144} = \frac{1}{24}.$$

This is the reciprocal of the time; and thus the time required for the conjoint filling is 24 minutes, as we found in the first or easy mode of statement, where the rates were explicitly specified among the data.

Another question of the same kind: If A can build a wall in 30 days, B in 40 days, and C in 50 days, how soon can they all build it, if they can all work together without interfering with each other?

Answer in x days, where

$$\frac{1}{30}+\frac{1}{40}+\frac{1}{50}=\frac{1}{x};$$

because, during each day, A does $\frac{1}{30}$th of the wall, B does $\frac{1}{40}$th, and C does $\frac{1}{50}$th; so the three together do, each day, what is represented by these fractions added together. Hence the number of days will be the reciprocal of the sum of these fractions.

It is probably undesirable to assist a beginner to so easy a solution of this class of problem prematurely, or until he has been afforded an opportunity of expending some thought upon it; for it is difficult to get a good grip of a thing which is too smooth and slippery.

CHAPTER XII.

Involution and evolution and beginning of indices.

BECAUSE $6 \times 6 = 36$, which may be called 6^2 (six square),
and $6 \times 6 \times 6 = 216$, and may be called 6^3 (six cube),
$6 \times 6 \times 6 \times 6 = 1296 = 6^4$, (six to the fourth power),
and so on,

it is customary to call 6 the **square root** of 36;
it is also the **cube root** of 216,
the **fourth root** of 1296,

and so on;
and the process of finding, or extracting, the root of any number is called evolution,—though the **name** is of small importance.

The **idea** of roots and powers however is of great importance and it is necessary to know how to find them.

The square root of 49 is 7; as we know from the multiplication table. So also we know in the same way, that is by direct experiment, that the square root of 64 is 8; because this is only another way of saying that 8 square, 8^2, or 8×8, equals 64.

The statement that $9^2 = 81$ is identical, in everything except in **form**, with the statement that the square root of 81 is 9.

The square root of 100 is 10,
that of 144 is 12,
and of 400 is 20.

A notation or mode of writing is necessary for roots, to avoid having constantly to write words, and for compactness; just as 3 is handier to deal with than "three," though it means the same thing.

The notation employed in involution or raising to powers we have already stated (p. 56), viz. little figures or indices placed after the main figure, as for instance $4^2 = 16$, the index denoting how many fours are to be multiplied together.

So 6^3 means that three sixes are to be multiplied together; and that is all that the **index** shows.

9^5 means that five nines are to be multiplied together; and the result is a big number, which a child may at once be set to calculate. He might also calculate such numbers as 2^2, 3^3, 4^4, 5^5, ..., 9^9, 10^{10}.

Moreover, he should at once write down the values of the following:

$$10^2,\ 10^3,\ 10^4,\ 10^5,\ 10^6,\ \ldots,$$

and perceive that in each case the number of ciphers following the *one* is indicated by the index. So he can write down in full 10^{25} without consideration, and can be told that the short form is a compact and handy and universally adopted method of expressing large numbers.

From all this, if a sharp child were asked to invent a notation for roots, he might perhaps, though it is much to expect if really ignorant of the convention, but he *might* suggest that since $4^2 = 16$, perhaps $16^{\frac{1}{2}} = 4$; or perhaps he might suggest 16^{-2} as a suitable notation. In either case he should be much encouraged.

Of the two notations, thus suggested, the first is correct and is employed. The second is employed for something also, but for something totally different from a root, viz. a reciprocal.

Let us get used to the notation for roots by fractional indices, and at the same time justify it as a consistent and convenient method.

First of all it must be admitted as not easy to put into words. The index 2 signifies that the number is to be raised to the second power, or multiplied by itself, so that

$$4^2 = 4 \times 4 = 16;$$

hence we might say that $16^{\frac{1}{2}}$ means that the number is to be raised to the half power, or multiplied,—how? It is hardly an interpretable phrase; so we must proceed more gradually.

First of all, it is simple to suppose that if the index is unity it should be understood to leave the figure unaltered, so that

$$4^1 = 4 \text{ and } 16^1 = 16;$$

therefore we may write indices on both sides, thus, $16^1 = 4^2$; let us next suppose that we may halve the index on each side getting $16^{\frac{1}{2}} = 4^1$, and read this, root 16 equals 4. We might halve the indices again, and get $16^{\frac{1}{4}} = 4^{\frac{1}{2}}$; which equals the square root of four, or 2; so that we may surmise that the fourth root of 16 is 2; and verify it thus,

$$2 \times 2 \times 2 \times 2 = 16.$$

Similarly,

$$27 = 3^3,$$
$$27^{\frac{1}{3}} = 3^1,$$

which agrees with the fact that the cube root of 27 is 3,

$$(27 = 3 \times 3 \times 3).$$

Again,

$$81^{\frac{1}{2}} = 9,$$
$$81^{\frac{1}{4}} = 9^{\frac{1}{2}} = 3.$$

All that we have here *assumed* (and it is a large assumption) is that in an equation involving terms with indices, if we perform an operation on the indices—provided we perform the same operation on both sides,—the equality remains undisturbed.

This is an assumption, a guess, an expectation, to be justified or contradicted experimentally by results. We shall find that its truth depends entirely on the kind of operation so performed. We happen to have hit first upon trying multiplication and division as applied to indices, and that seems to work correctly. But we shall try other operations shortly and will find them fail.

Those who imagine or assert that **experiment** has no place in mathematics do not know anything about mathematics. Sometimes results are arrived at by theory, sometimes by experiment, sometimes by a mixture of the two;—either theory first and confirmation by experiment, or experiment first and justification by theory: just as in Physics or any other developed science.

Let us now press our assumption to extremes and experiment on it in various ways so as to see whither it will lead us. Start with any equation, such as

$$4^2 = 16^1.$$

Double each index, and we get

$$4^4 = 16^2 = 256.$$

So the fourth root of 256 is given as 4, and the eighth root will accordingly be 2, or 256 is asserted to be the eighth power of two; which is the fact: eight twos multiplied together do yield 256.

Treble the index, and it becomes

$$4^6 = 16^3 = 256 \times 16 = 4096,$$

or conversely, $4096^{\frac{1}{6}} = 4$.

Hence the sixth root of 4096 is given as 4, and

$$4096^{\frac{1}{12}} = 4^{\frac{1}{2}} = 2,$$

that is, its twelfth root is 2. Again fact agrees with theory. 2 multiplied by itself 12 times does equal 4096.

Hence it appears that the operation of multiplying indices by any the same factor on each side of an equation may be trusted to give true results.

So also division of indices by any the same number may be trusted too; thus starting as before with

$$16^1 = 4^2,$$

quarter each index $\qquad 16^{\frac{1}{4}} = 4^{\frac{1}{2}} = 2,$

halve each again $\qquad 16^{\frac{1}{8}} = 4^{\frac{1}{4}} = 2^{\frac{1}{2}}.$

How are we to calculate $2^{\frac{1}{2}}$? That is not an easy matter: we will leave it unvalued for a time and merely call it the square root of 2. It is often denoted by a sort of badly-written long-tailed r in front of the digit, thus, $\sqrt{2}$ or $\sqrt{2}$.

Try again, $\qquad 27 = 3^3,$

$$27^{\frac{1}{3}} = 3,$$

$$27^{\frac{1}{6}} = 3^{\frac{1}{2}},$$

there is the same difficulty about interpreting $3^{\frac{1}{2}}$; no whole number will serve. We can call it the square root of 3, or briefly "root 3," and can denote it by writing $\sqrt{3}$ as before.

$\sqrt{16}$ means the same as $16^{\frac{1}{2}}$, namely 4; and $\sqrt{4} = 4^{\frac{1}{2}} = 2$; but whereas the fractional index contains an important and valuable idea, which remains to be developed, the symbol $\sqrt{\ }$ is nothing but shorthand for the word "root," and is itself trivial and inexpressive, though quite harmless and of constant service.

What we have learnt from the above examples resulting in $\sqrt{2}$ and $\sqrt{3}$ is that when employing fractional indices we can arrive at something, easy of interpretation indeed, but not easy of numerical evaluation; there is no need to mistrust the result but only to wait till more light can fall upon it.

Now try some other operations applied to indices, we shall find that wariness is necessary, and that mere guesses and surmises as to what it is permissible to do to equations are not worth much. Everything must be tested. Suppose we try squaring them on both sides as thus: Starting with

$$4^2 = 16^1,$$

squaring indices would give us

$$4^4 \not= 16^1,$$

since the square, or any other power, of 1 is 1,

$$1 \times 1 \times 1 \times 1 = 1.$$

The result, that 16 is both the square and the fourth power of 4, is false and absurd: and hence the sham equation is erased.

So we learn that whereas multiplication of indices by any factor is an operation that can be trusted to give true results, and division of indices by a factor can probably be trusted too, since one operation is the inverse of the other, yet that involution is not an operation that can legitimately be performed upon indices, but only upon the numbers themselves.

Suppose we try addition, equal additions to the indices on each side; add 1 for instance, we get $4^3 \not= 16^2$, which is a falsehood if the equality sign is left unerased.

It is time we began to consider what operations are really legitimate and what are not; and gradually in both cases we must proceed to ask, Why?

CHAPTER XIII.

Equations (*treated by the method of very elementary experiment*).

It is therefore convenient at this stage to introduce the idea of an expressed equality, which is called an equation, and to consider what are the operations to which an equation can be subjected without destroying the equality.

It is customary to postpone this subject to Algebra, but we do not wish to perpetuate any sharp distinction between algebra and arithmetic, and it is useful to begin experimenting with equations while still they are expressed in terms familiar to beginners.

Typical equations are of many kinds, of which we may now consider the following:

The addition kind,	$3 + 2 = 5$.
The subtraction kind,	$3 - 2 = 1$.
The multiplication kind,	$3 \times 2 = 6$.
The division kind,	$3 \div 2 = 1{\cdot}5$.
The involution kind,	$3^2 = 9$.
The evolution kind,	$9^{\frac{1}{2}} = 3$.

There are plenty of others, but these will do to begin with. Every equation has two sides, called respectively the left-hand side and the right-hand side; the symbol = is the barrier. It is not an impassable barrier, but terms get reversed when they are taken across it; positive becomes negative, and *vice versa*. In order to find out what may be done to equations we can experiment.

Take any of these equations and try experiments on it. For instance, add something to or subtract something from each side. So long as we add the same thing to each side no harm is done: the equality persists. For instance, start with the first two of the above equations and modify them by addition or subtraction in various ways:—

$$3+2+7 = 5+7 = 12;$$
$$3-2-1 = 1-1 = 0;$$
$$3+2-6 = 5-6 = -1,$$
$$3-2+a = 1+a,$$
$$x+3-2-a = x-a+1,$$
$$3-2+2 = 1+2=3,$$
$$3+2+\tfrac{1}{2} = 5\tfrac{1}{2}.$$

So far everything is very simple and safe.

Not only may we add the same thing to each side, but we may add equal things to each side (which may be regarded as an illustration of the axiom, that if equals be added to equals the wholes are equal).

Thus $\left.\begin{array}{l}3+2 = 5,\\ 7+6 = 13.\end{array}\right\}$ and also

So $3+2+7+6 = 5+13 = 18.$

Or again, $\left.\begin{array}{l}3-2 = 1,\\ 3^2 = 9.\end{array}\right\}$ and

So $3-2+3^2 = 1+9 = 10.$

Also take the following:

$$\left.\begin{array}{ll} & 3-2 = 1, \\ \text{and} & 5-6 = -1; \end{array}\right\}$$

$$\therefore\ 3-2+5-6 = 1-1 = 0.$$

But take an equation of the multiplication kind,

$$3\times 2 = 6,$$

a little caution is necessary in adding anything to the left-hand side.

We might have $(3 \times 2) + 1 = 6 + 1 = 7,$
or we might have $3 \times (2 + 1) = 3 \times 3 = 9.$

If we only write $3 \times 2 + 1$, without brackets, it is ambiguous; for the value depends on whether the addition or the multiplication is performed first: that is, on whether the 2 is grouped along with the 3 or with the 1, but the brackets enable us to indicate the grouping clearly.

Take another example,

$$7 \times 8 = 56,$$

$$(7 \times 8) - 4 = 52,$$

although

$$7 \times (8 - 4) = 28;$$

but the last is quite a different equation, and is not deduced by simple subtraction of 4 from both sides.

About the other forms of equations there is no difficulty; we will just write them, with something either added to or subtracted from each side:

$$\tfrac{3}{2} - 1 = 1{\cdot}5 - 1 = {\cdot}5 = \tfrac{1}{2},$$

$$3^2 - 1 = 9 - 1 = 8,$$

$$9^{\frac{1}{2}} + 2 = 3 + 2 = 5.$$

Incidentally we here observe the advantage of the fractional notation over the ÷ notation. If we had written $3 \div 2 - 1$ we should have had to avoid ambiguity by the use of brackets, as was necessary in multiplication; but $\frac{3}{2} - 1$ is unambiguous. Unity is subtracted from the whole fraction, not from either numerator or denominator. If unity were subtracted from the numerator it would not be right,

$$\frac{3-1}{2} \neq 1{\cdot}5 - 1 = {\cdot}5;$$

nor will it do to subtract from the denominator, nor from both.

So much at present for addition and subtraction; now try multiplication and division: start with

$$3 + 2 = 5;$$

double each term, $6 + 4 = 10$;

treble each term, $9 + 6 = 15$;

halve each of these terms,

$$4\tfrac{1}{2} + 3 = 7\tfrac{1}{2}.$$

So here we are safe.

Proceed now to the factor or multiplication form of equation:

$$3 \times 2 = 6;$$

double each digit, $6 \times 4 \neq 12$,

and we get wrong.

We learn that we must not double each factor in a product, though we must double each term of a sum; hence the expression $3 + 2$ is commonly spoken of as containing two terms, but 3×2 is spoken of as a single term.

To double the single term it is sufficient to double *one* of its factors; so if we write

$$6 \times 2 = 12$$

we get right again.

Similarly we must halve one of the factors only,

$$1\tfrac{1}{2} \times 2 = 3,$$

or else $3 \times 1 = 3$.

Now attend to the quotient form,

$$\tfrac{3}{2} = 1{\cdot}5;$$

double every digit, $\tfrac{6}{4} \neq 3$,

and we get wrong.

Double the denominator only,

$$\tfrac{3}{4} \neq 3,$$

it is still wrong.

Double the numerator only, $\tfrac{6}{2} = 3$,

and we get right.

So in multiplication and division of a quotient by a whole number, the factor has to be applied to the numerator only.

Take another example, $\frac{12}{3} = 4,$

$$\frac{6}{3} = \frac{4}{2} = 2,$$

$$\frac{3}{3} = \frac{4}{4} = 1,$$

$$\frac{24}{3} = 4 \times 2 = 8.$$

Finally, take the involution form,

$$3^2 = 9.$$

What are we now to double if we want to double both sides?

$$6^2 \neq 18 \text{ is wrong,}$$

$$3^4 \neq 18 \text{ is also wrong.}$$

We cannot do it quite so simply; so we must write merely

$$2 \times 3^2 = 18,$$

which leaves the step really undone and only indicated.

But take another example,

$$3 \times 3^2 = 27.$$

This could be written 3^3.

Again, $3^3 \times 3^2 = 27 \times 9 = 243 = 3^5$,

and a rule of extraordinary interest and usefulness is suggested.

Think it over, we shall return to it in Chapter XVI.

Further consideration of what can be done to equations.

A sentence like the following:

"If both sides of an equation be treated alike, the equality will persist," might easily be considered axiomatic; but so much caution is required before we can be sure that both sides have been really treated alike, that it is highly dangerous to employ such an axiom. We have already come across some cases of the danger, but the subject is very important and will bear fuller treatment.

The general doctrine may be laid down that before we understand properly what can be done, or what it is permissible to do, in any subject whatever, we should take pains to ascertain also what cannot be done under the same circumstances, *i.e.* what it is not possible to do without error. This latter part should not be too long dwelt upon, because error is most simply excluded by attention to and familiarity with the correct processes, so that presently all others instinctively feel wrong; but once at least we should examine the whole matter, and learn, if we can, why one set of things are wrong and another set right. This remark applies also to other things than arithmetic.

An equation consists of two sides, and each side consists of terms. Frequently the right hand side is zero, especially in algebra and in higher mathematics. Sometimes, instead of being zero, it is some constant or other independent quantity, and is called "the absolute term," because it is undetermined by anything on the left hand side: to which however it is equated.

An equation is the most serious and important thing in mathematics. The assertion that two quantities or two sets of quantities are equal to each other, whether it is meant that they are always equal, or only that they are equal under certain circumstances which have to be specified, is a very definite assertion and may carry with it extraordinary and at first unsuspected consequences.

The equations we are now using as illustrations are by no means of this high character; they are usually mere identities, and depend on the truism that a combination of things grouped or expressed in one way are unchanged in number when grouped or expressed in another way.* But although it may

* We call this a truism; but it is a dangerous term to employ, and when we come to Chemistry we must be on our guard against assuming that the volume of H_2+O is equal to that of H_2O, under the same external circumstances. It is true of weights (as nearly as we can tell) but it is not even approximately true of volumes.

be some time before they realise the vast importance attaching to equations, children will take it on trust that they are now entering the central arcana of the subject, and will be willing to give the needful attention to the processes which have constantly to be employed. An initial account of them is given in the following chapter, parts of which may be read before the whole of the following introductory matter.

When a number of quantities are multiplied together, they are held to constitute one term. Whenever the sign + or − intervenes, it interrupts the term, and each such sign has a term on either side of it.

Thus $a+b$, $70-6$, are each two terms; but ab, and 70×6, and abc, and 10^6, and $5a\surd 2$, and $\frac{10ab}{x}$ are all single terms.

What about such an expression as

$$\frac{5ab-x}{abx},$$

where there is one (or more than one) addition or subtraction sign in the numerator?

Answer: So long as it is kept all together it can be called one term, but it can easily be split into two, viz.

$$\frac{5}{x}-\frac{1}{ab},$$

and for some purposes its terms can be considered plural without re-writing.

The long line of division in the original expression however may be held to weld the whole into one term; and brackets have the same effect. Thus,

$$(a+b),\ (70-6),\ 5(a+b)x,\ \surd(ax-by)$$

are all single terms once more; until the brackets are removed. And removal of brackets is an operation to be performed cautiously. Rubbing them out is not a legitimate way of removing them.

For instance, $3(7-4) = 9$;
but $37-4$, and $3\times 7-4$, and $3\cdot 7-4$
are all different.

Again, $\sqrt{(16+9)} = 5$,
but $\sqrt{16}+\sqrt{9} = 7$,
and $\sqrt{16}+9 = 13$,
while $16+\sqrt{9} = 19$;
the three are entirely distinct statements from the first, and are not deducible from it.

So we learn that the right removal of brackets is a matter to be studied.

When we assert that the same operation can properly be applied to each side of an equation then, we must be careful to interpret it always as an operation applied to the whole side, and not to any part of it. We may not tamper with one term and leave the others alone, nor must we tamper with a part of a term only. Nor must we repeat the operation for each of the factor components of a single term.

This must be illustrated:

Given that $a+b = c$,
it is correct to say that
$$2a+2b = 2c,$$
or that $2(a+b) = 2c$:
but given that $a\times b = c$,
it is not correct to say that
$$2a\times 2b = 2c.$$
For here the term ab is one, and it only needs doubling once.

Given that $a^2 = b^3$,
it is true that $5a^2 = 5b^3$, ...(1)
but it is not true that $(5a)^2 = (5b)^3$, ...(2)
for that would mean $25a^2 = 125b^3$,
which, subject to the given data, is absurd, unless a and b are both zero.

In reading the two lines labelled (1) and (2) it is customary to read them carefully in order to discriminate what otherwise would sound quite similar. The former of the two lines is read

"five a-square = five b-cube";

the latter of the two lines is read

"five a, squared = five b, cubed";

and these are quite different. They cannot under any circumstances be both true (unless indeed a and b are both zero). They are therefore called "inconsistent" equations

(like $x = y$, and $x = 2y$, which cannot *both* be true).

To illustrate the inconsistency, take an example:

$8^2 = 4^3$, both being 64,

and so also $5 \times 8^2 = 5 \times 4^3$, both being 320,

but $(5 \times 8)^2 \neq (5 \times 4)^3$, the one being 1600 and the other 8000.

Read the sign $\neq$ as "does not equal."

Given again that $a^2 = b^3$,

it does not follow that $a^3 = b^4$,

although we have done the same thing, that is added 1, to the index on each side.

Nor would it be true to say that

$$a^4 = b^9,$$

although we have now squared the index on each side.

But it does turn out true that if we double or treble the index, the equality persists: given that $a^2 = b^3$

it is true that $a^4 = b^6$

and that $a^6 = b^9$,

so that it appears as if it were permissible to multiply the index on each side by any the same factor. We must examine this later, but at present we will merely verify the truth of these last assertions by an arithmetical example:

For instance $$8^2 = 4^3,$$
whereas $$8^3 \neq 4^4,$$
one being 512, and the other 256 ;
but $$8^4 = 4^6,$$
both being $(64)^2$ or 4096.

Likewise $$8^6 = 4^9,$$
both being $(64)^3$ or 262,144.

Now take a slightly more general type of equation.
$$ax = by,$$
it follows that $$(ax)^3 = (by)^3,$$
but it is by no means necessary that ax^3 shall equal by^3.

For instance, $$7 \times 4 = 14 \times 2,$$
and $$(7 \times 4)^3 = (14 \times 2)^3;$$
but $$7 \times 4^3 \neq 14 \times 2^3,$$
for one equals 448, and the other equals 112.

Take, as given, the equation $ab = xy$, and let us multiply, add, and divide on both sides, so as to illustrate legitimate and illegitimate operations; the pupil being left to devise numerical illustrations and tests for himself.

First multiply or divide by any quantity whatever, say c.
$$abc = cxy;$$
or, assuming another quantity $z = c$, we may write it
$$abc = xyz.$$

So also $$\frac{ab}{c} = \frac{xy}{z},$$
$$a \times b \div c = x \times y \div z = y \times x \div z,$$
$$\frac{1}{c}(ab) = \frac{a}{c} \times b = a \times \frac{b}{c} = x \times \frac{y}{z} = y \times \frac{x}{z} = \frac{1}{z}(xy).$$

Next add or subtract something to or from each side.
$$ab + c = xy + c,$$
$$ab - b = xy - b \neq xy - y,$$
$$ab - x = xy - x,$$
$$ab - xy = xy - xy = 0.$$

This last is worth attention. The result has been to transfer a term from one side of the equation to the other, its sign being changed in the process. This is important and demands further illustration.

Let $$x = 6.$$

We can equally well write it, by subtracting 6 from both sides, $$x - 6 = 0,$$
and the 6 has been transferred, with change of sign.

Or let $$x = y.$$

Subtract y from both sides,
then $$x - y = 0.$$

Again let $$x = -y,$$
then add y to both sides and we get
$$x + y = 0.$$

Or let $$ax + by = -c,$$
add c to both sides, $ax + by + c = 0.$

This kind of simple operation has constantly to be performed.

One more illustration therefore:

Let $$ax + by = cx + dy.$$

We can subtract the right hand side from both sides; in other words, transfer it to the left, with change of sign; getting $$ax + by - cx - dy = 0,$$
which is more neatly written
$$(a - c)x + (b - d)y = 0;$$
or again, $$(a - c)x = (d - b)y.$$

Here the last mode of expression is deserving of attention. We will arrive at it more directly.

To this end start again with
$$ax + by = cx + dy;$$
transfer by to the right, and cx to the left; thus we get
$$ax - cx = dy - by,$$

or what is the same thing,

$$(a - c)x = (d - b)y.$$

Divide each side by the product $(a - c)(d - b)$ and the equation becomes

$$\frac{(a - c)x}{(a - c)(d - b)} = \frac{(d - b)y}{(a - c)(d - b)}.$$

In each of these terms there is a common factor in numerator and denominator, so we can cancel them, and are left with

$$\frac{x}{d - b} = \frac{y}{a - c}.$$

Or we might have divided otherwise, and arrived at any of the following:

$$\frac{x}{y} = \frac{d - b}{a - c}$$

$$x : y = (d - b) : (a - c),$$

$$y : x = (a - c) : (d - b),$$

$$x = \frac{d - b}{a - c}y,$$

$$y = \frac{a - c}{d - b}x,$$

$$\frac{(a - c)x}{(d - b)y} = 1,$$

$$\frac{(a - c)x}{(d - b)y} - 1 = 0,$$

$$\frac{a - c}{d - b} - \frac{y}{x} = 0.$$

All these are entirely equivalent forms; and as an exercise they should all be deduced, any one from any other.

And all can be numerically illustrated, by attributing to the symbols some particular values; for instance by taking

$$x = 24,\ y = 2,\ a = 5,\ b = -19,\ c = 1,\ d = 29.$$

Let this be done, as an exercise.

Among things that can legitimately be done to equations are certain operations which are by no means obvious, and demand attention.

Suppose we are told that

$$\frac{x}{y} = \frac{a}{b} \quad \text{.......................(1)}$$

We are not allowed to say $\frac{x-1}{y} = \frac{a-1}{b}$; but we are entitled to say that

$$\frac{x-y}{y} = \frac{a-b}{b}, \quad \text{.......................(2)}$$

because this is equivalent to subtracting unity from both sides, *i.e.* is equivalent to

$$\frac{x}{y} - 1 = \frac{a}{b} - 1.$$

So also we might have truly written

$$\frac{x+y}{y} = \frac{a+b}{b}. \quad \text{.......................(3)}$$

But from the truth of these two operations it follows that we might also have written

$$\frac{x-y}{x+y} = \frac{a-b}{a+b}. \quad \text{.......................(4)}$$

For this would be obtained if we had divided each side of equation (2) by the corresponding side of equation (3); for if equals be divided by equals the quotients are equal.

Let us illustrate this important result arithmetically.

Start with $\frac{14}{6} = \frac{49}{21}$, which can be easily proved true, and may be taken as corresponding to (1).

Then it follows that

$$\frac{14-6}{6} = \frac{49-21}{21};$$ in other words $\frac{8}{6} = \frac{28}{21}$; which corresponds to the form numbered (2);

also, like (3), that $\frac{14+6}{6} = \frac{49+21}{21}$, *i.e.* that $\frac{20}{6} = \frac{70}{21}$;

and, like (4), that $\frac{14+6}{14-6} = \frac{49+21}{49-21}$, or that $\frac{20}{8} = \frac{70}{28}$.

Or each member of any of them may be inverted: for instance the last:
$$\frac{8}{20} = \frac{28}{70}.$$

Starting once more with
$$\frac{x}{y} = \frac{a}{b}$$
we might equally well write it
$$\frac{x}{a} = \frac{y}{b}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$
for this is the result of multiplying both sides by $\frac{y}{a}$; so therefore it is true to say that
$$\frac{x-a}{a} = \frac{y-b}{b}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$
and
$$\frac{x+a}{a} = \frac{y+b}{b}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$
also that
$$\frac{x-a}{x+a} = \frac{y-b}{y+b}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$
and that is really, though by no means obviously, the same thing as equation (4).

Illustrate this too, numerically, with the same numbers as before:
$$\frac{14}{49} = \frac{6}{21} \quad \text{corresponds to (5)}$$
$$-\frac{35}{49} = -\frac{15}{21} \quad \text{corresponds to (6)}$$
where the minus signs may equally well be omitted or cancelled by multiplying each side by -1;

and
$$\frac{35}{63} = \frac{15}{27} \quad \text{corresponds to (8)}$$

or again any of them may be inverted, *e.g.*:

$$\frac{49}{14} = \frac{21}{6}, \text{ etc.}$$

Now let us apply the so-called **involutional** operations to both sides of an equation, and ascertain what we may do and what we may not do.

Begin with $$ab = xy.$$

Square both sides, $$(ab)^2 = (xy)^2.$$

Square each factor, $$a^2b^2 = x^2y^2.$$

Square one of them only, and we get wrong,

$$ab^2 \neq xy^2.$$

Take the square root of both sides,

$$\sqrt{(ab)} = \sqrt{(xy)},$$

also of each factor, $$\sqrt{a}\sqrt{b} = \sqrt{x}\sqrt{y},$$

or what is the same thing,

$$a^{\frac{1}{2}}b^{\frac{1}{2}} = x^{\frac{1}{2}}y^{\frac{1}{2}}.$$

So far we are all right except in the one marked instance: as can be tested by giving suitable numerical values to the four symbols.

But now take an equation with more than one term on a side, say $$x + y = c.$$

Square both sides $$(x+y)^2 = c^2.$$

Square each term, $$x^2 + y^2 \neq c^2,$$

and we get wrong.

This is a mistake constantly being made by beginners, and it must be further emphasised. As an example,

$$4 + 5 = 9,$$

$$(4+5)^2 = 9^2 = 81,$$

but $$4^2 + 5^2 = 16 + 25 = 41 \neq 9^2.$$

The following fallacy may serve as an illustration:

$$\sqrt{25} = \sqrt{(16+9)}, \quad \therefore\ 5 = \sqrt{16} + \sqrt{9} = 7.$$

Observe that these numerical instances, if they lead to error, show quite decidedly that the operation tested is wrong. They do not prove with equal validity that it is right, if they turn out correctly: certainly a *single* instance of correctness is insufficient. They render its rightness probable, but the *rationale* of it has to be further investigated. A single instance of real error however is sufficient to invalidate any operation under test.

Exercises.—Test the correctness of the following horizontally juxtaposed statements :

$$
\begin{array}{rcl@{\qquad}rcl}
2 \times 3 &=& 6. & 2 + 3 &=& 5. \\
2^2 \times 3^2 &=& 6^2. & 2^2 + 3^2 &\neq& 5^2. \\
5 \times 6 &=& 30. & 5 + 6 &=& 11. \\
5^2 \times 6^2 &=& 30^2. & 5^2 + 6^2 &\neq& 11^2. \\
2^3 \times 6^3 &=& 12^3. & 2^3 + 6^3 &\neq& 12^3. \\
 & & & 2^3 + 6^3 &\neq& 8^3. \\
4 \times 9 &=& 36. & 4 + 9 &=& 13. \\
4^{\frac{1}{2}} \times 9^{\frac{1}{2}} &=& 36^{\frac{1}{2}} = 6. & 4^{\frac{1}{2}} + 9^{\frac{1}{2}} &=& 5. \\
9 \times 144 &=& 1296. & 9 + 144 &=& 153. \\
\sqrt{9} \times \sqrt{144} &=& \sqrt{(1296)}. & \sqrt{9} + \sqrt{144} &=& 15. \\
27 \times 216 &=& 5832. & 27 + 216 &=& 243. \\
27^{\frac{1}{3}} \times 216^{\frac{1}{3}} &=& 5832^{\frac{1}{3}}. & 27^{\frac{1}{3}} + 216^{\frac{1}{3}} &=& 9. \\
\text{or } \ 3 \times 6 &=& 18. & & &
\end{array}
$$

But now it must be admitted that this experimental mode of treatment may not be considered the best mode of beginning the experience of equations: and it is certainly not the most conducive to rapid progress; it may be better therefore to apply treatment like that of the present chapter at a rather later stage and to use it as a cautionary and salutary exercise. The importance of the subject is so great that it can hardly be over-emphasised, nor is one mode of approach sufficient. In the next chapter a somewhat more orthodox and quite effective mode of procedure is adopted.

CHAPTER XIV.

Another treatment of Equations.

EQUATIONS may be classified in various ways: there are such things as differential equations, there are quadratic equations and equations of the fifth degree, etc., but for the present we will classify them under three simple heads:—

1st. Statements of specific or particular fact, such as:

$$3+4 = 7,$$

or

$$9\sqrt{(144)} = 108;$$

these involve only definition and re-grouping.

2nd. Statements of general or universal truth, such as:

$$n^2-1 = (n+1)(n-1)$$

$$\log a^x = x \log a;$$

these are called identities, and are frequently denoted by a triple sign of equality $\equiv$, for instance $a+b \equiv b+a$, whenever it is desired to emphasise their distinction from the third class.

3rd. Equations proper, or statements of condition or information such as:

$$17x = 34,$$

or

$$5x^3 = 40;$$

statements which are not by any means generally true, but are only satisfied by some implicit datum, such as, in the above instances, $x=2$.

4th. There is also, from this point of view, a fourth class of equations, expressive of a relation between two quantities, such as

$$3x + 4y = 12,$$

or

$$x^2 + y^2 = 25;$$

which are satisfied not by all possible values of x and y, as an identity is satisfied, but by an exclusive and definite though infinite series of values.

The first is satisfied, on a certain geometrical convention, by all the points which lie on a specific straight line, the second by all the points which lie on a definite circle.

If the equations are given simultaneously, they are satisfied together by two and only two points, viz. the points where the straight line cuts the circle.

With this fourth class we have nothing to do just yet: it opens up a large and exhilarating subject.

With the first kind of equation we have constantly had to do already: all purely arithmetical equations are necessarily of this kind.

The second kind is constantly encountered throughout algebra and trigonometry; identities represent the skeleton or framework of mathematical science, all its universal and undeniable truths can be thus expressed.

The third kind of equation, or equation proper,—equations which have a definite solution, equations which convey specific information about an unknown quantity, and express it in terms of numbers or known quantities of some kind—those are the equations with which we deal in this chapter, and that is the kind which gives immediate practical assistance towards the solving of problems.

The process of "solving" an equation is simply the act of reducing it to its simplest possible form. Written in any form the equation conveys the same information, but in some forms it is not easy to read; the solving of it is analogous to the interpretation of a hieroglyph or the translation of an unknown phrase.

For instance the following equations

$$7x - 2\tfrac{1}{2} = \tfrac{1}{2}x + 30$$
$$13x = 65$$
$$x = 5.$$

all express the same fact and convey the same information concerning x, but the last obviously conveys it in simplest form, and it is called the "solution" of the first, the second being an intermediate step.

The two sides of an equation may be likened to the two pans of a balance, containing equal weights of different materials or of the same material differently grouped. It is permissible to take from or to add equal quantities to both pans, the balance or equality being still preserved; but a weight must not be taken out of one pan and added to the other, unless its force be reversed in direction and made to act upwards instead of downwards; which can be actually managed by hanging it to a string over a fixed pulley, the other end of the string being attached to the pan.

This fact is most simply expressed by saying that if any term or quantity is transferred from one side of an equation to the other, it must be reversed in sign, if the equality is still to persist, *i.e.* if the equation is to remain true.

This is a simple but important matter of constant practical use, and it requires illustration:

Let the equation be given

$$x - 2 = 3.$$

We can transfer -2 to the other side of the equation, where it will become $+2$, giving us $x = 3 + 2$ or in other words $x = 5$. (We may consider that we have added 2 to each side.)

The value 5 obviously satisfies the equation in its original form, because it is true that $5 - 2 = 3$; and the substitution of the found value in the original equation and then seeing if it fits or holds good, is called 'verifying' the solution.

Take another case:

$$3x^2 + 17 = 4x^2 - 8.$$

Getting the unknown quantities on one side, and the known on the other, it becomes

$$3x^2 - 4x^2 = -8 - 17;$$

if we like we may now reverse the sign of every term, which will give us

$$4x^2 - 3x^2 = 17 + 8$$

or

$$x^2 = 25$$

or

$$x = 5.$$

Thus all the equations we have written recently happen to be expressive of the same fact: namely that the particular x denoted by them is merely the number 5. Substituting this number in the above equation, it becomes

$$3 \times 25 + 17 = 4 \times 25 - 8$$

or

$$75 + 17 = 100 - 8$$

which we perceive to be an arithmetical identity, since both sides $= 92$, thus the solution is verified, or the value $x = 5$ is proved to satisfy the given equation.

We do not know for certain that it is the only value that will satisfy it, but at any rate it is *one* solution. It so happens that the equation last written will also be satisfied by the solution $x = -5$; and this is characteristic of square or quadratic equations in general, that there are two answers instead of only one.

An equation of the third degree, that is an equation involving x^3, will in general have three answers; and so on.

Take one more quite simple example, for practice:

given $7x - 12 = 5x + 6$, to find x.

Subtract $5x$ from both sides, or, what is equivalent, transfer $5x$ over to the other side with change of sign,

we get $$2x - 12 = 6.$$

Now add 12 to both sides, or, what is the same thing, transfer -12 over to the other side, and it becomes

$$2x = 6 + 12 = 18;$$

wherefore $$x = 9 \text{ is the solution.}$$

Try it in the original equation in order to verify it and we get $$63 - 12 = 45 + 6,$$

which is an arithmetical identity.

As to algebraic identities, it is probably needful to remind young beginners occasionally even of such simple facts as these: at the same time making mysterious hints that there are possible interpretations, to be met with hereafter, wherein even these simple statements lack generality and are open to reconsideration, $$a + b \equiv b + a,$$

and $$ab \equiv ba;$$

and they should be frequently reminded of such useful identities as

$$(a + b)^2 \equiv a^2 + 2ab + b^2,$$
$$(a - b)^2 \equiv a^2 - 2ab + b^2,$$
$$(a + b)(a - b) \equiv a^2 - b^2.$$

Oral questions should be asked at odd times concerning equivalent expressions for such things as

$(p + q)^2$,	$(x - b)^2$,	$(x + y)(x - y)$,
$(x + 1)(x - 1)$,	$(n + 1)^2$,	$(n - 1)^2$,
$(1 + a)^2$,	$(1 - a)^2$,	$(x + 3)^2$,
$(x - 5)^2$,	$(3 + \sqrt{2})^2$,	$(7 + \frac{1}{2}x)^2$, etc., etc.,

since a pupil's knowledge of such fundamental things should be ready for immediate application—like a well constructed machine.

We have not yet taken an example of an equation involving x^2 as well as x, because they are not quite so easy to solve;

but a parenthetical remark may be introduced even at this stage. We know that quantities of different kinds do not occur in one expression; in other words, that all the terms of an expression must refer to the same sort of thing, if they are to be dealt with together or equated to any one value. Nevertheless an expression like x^3+5x^2+2x+6 is common, and x may be a length; which looks as if we could add together a volume, an area, a length, and a pure number. Not so, really, however: see Appendix.

The equation $(x-3)(x-4) = 0,$

written out, becomes $x^2-7x+12 = 0,$

or $x^2 = 7x-12.$

We may guess at numbers which will satisfy this equation, and we have been told there must be two, because it is a quadratic: it contains x^2. By trial and error it will be found that the number 3 and the number 4 will both satisfy it; for insertion of the first gives the identity $9 = 21-12$, and insertion of the second gives the identity $16 = 28-12$; but no other number whatever, when substituted, will result in an identity, that is to say no other number will satisfy the equation; the equation has two, and only two, solutions, or, as they are often called, "roots."

Looking at the factor form of the equation with which we started, $(x-3)(x-4) = 0,$

it is *obvious* that either 3 or 4 will satisfy it; because the value 3 makes the first factor zero, and the value 4 makes the second factor zero It is not necessary that both factors shall be zero —either will do—hence the useful answer is not necessarily *both* 3 and 4, but either 3 or 4, or possibly both.

The factor form of writing the equation, therefore, contains the solution in so obvious a manner, that it is sometimes spoken of as "the solution": and if an equation like

$$3x^2+7x-31 = 11-8x$$

were, by any process of manipulation, reduced to the form

$$(x-2)(x+7) = 0$$

it would be considered solved; because it is then obvious that the values $+2$ and -7, that is to say either $x = 2$ or $x = -7$, or both, satisfy the equation. Inserting them successively, for the purpose of verification, we get for the value $x = 2$

$$12+14-31 = 11-16$$

which is an identity;
and for the value $x = -7$

$$147-49-31 = 11+56$$

which is another identity.

In collecting the terms of the given equation the two x terms can be put together, making $15x$, and the two absolute terms can be put together, making 42, but neither of these pairs can be merged in the other, nor in the term $3x^2$; there are essentially three distinct kinds of terms in the equation, and they must be kept distinct.

Introduction to Quadratics.

When beginning quadratic equations, it is a good plan to give them first of a kind that can easily be resolved into simple factors, so as to remove the appearance of difficulty, and yet to suggest a real method of solution.

For instance, $$x^2-7x+12 = 0$$
has roots 3 and 4, for these numbers add together to 7 and multiply together to 12. So the expression on the left hand side can be resolved into factors as

$$(x-3)(x-4)$$

and the equation can be re-written

$$(x-3)(x-4) = 0.$$

Again $$x^2-5x+6 = 0$$
is plainly satisfied by the values $x = 3$ and $x = 2$.

Once more, $x^2 - 11x + 30 = 0$
has the roots 5 and 6, and is equivalent to

$$(x-5)(x-6) = 0.$$

If we had chosen the equation

$$x^2 + 11x + 30 = 0$$

the roots would have been -5 and -6, and the equation written in the factor form would have been

$$(x+5)(x+6) = 0.$$

And so on, according to innumerable examples given in every text book of algebra.

When a quadratic expression equated to 0 is solved, it is always really resolved into two factors, for it is always virtually expressed in the form

$$(x-a)(x-b) = 0,$$

where a and b are the two numbers which satisfy the equation, its two "roots" as they are called; a term which is thus used in a new sense, having no reference to square or cube root.

Multiplying out the above expression, it takes the form

$$x^2 - (a+b)x + ab = 0$$

so that the coefficient of the middle term is the sum of the roots, and the absolute term is their product; provided that the coefficient of the quadratic term is unity, and the sign of the middle term is negative.

The process of solving the equation is the same as that of resolving the above expression into factors, and one way of achieving it is to think of two numbers which add together into the middle term and multiply together into the absolute term, provided the coefficient of the quadratic term is unity.

Suppose, for instance, the equation given were a general quadratic in x,

$$Ax^2 + Bx + C = 0.$$

Divide everything by A in order to reduce the coefficient of the quadratic term to unity, getting

$$x^2 + \frac{B}{A}x + \frac{C}{A} = 0.$$

Here we know that the sum of the roots of the equation must be equal to the ratio $-B/A$ and that the product of the roots must be equal to the ratio C/A. (See also Appendix III.)

In the above cases it was easy to guess the roots, but it is by no means always easy. A process must be used for finding them, and as far as possible the pupil should be left to find it out: with guidance, but no more actual telling than may be found necessary. But time and perseverance will be required. If the child has no head for it the attempt may be useless, and should not be persisted in unduly; nor should any disgrace attach to failure; success is a triumph rather than otherwise.

If the equation $x^2 + 10x = 24$ is given, it happens to be rather obvious that 12 and 2 must be the numbers concerned, if the signs are properly attended to; but the rule for finding them in general will have to be evolved from a consideration of the chief quadratic identity,

$$(x+a)^2 = x^2 + 2ax + a^2.$$

Suggest a trial of this to the pupil, and if necessary suggest trying the value 5 for the auxiliary and gratuitously introduced symbol a, because that will give

$$x^2 + 10x + 25,$$

which imitates the only hard part, the left-hand side, of the given equation: for a little complication of the easy, the numerical, part on the right-hand side, does not matter. So we might write the given equation now if we like,

$$x^2 + 10x + 25 = 24 + 25 = 49;$$

but directly we have done that, the equation is practically solved, for it is plainly equivalent to

$$(x+5)^2 = 7^2,$$

and therefore to $$x+5 = \pm 7;$$

that is to say x = either 2 or -12, as the case may be; for either will satisfy or solve the equation.

Wherefore the given equation, with the roots wrapped up

$$x^2+10x-24 = 0,$$

may likewise be written $(x-2)(x+12) = 0$, with the roots visible.

Another example or two to clinch the matter:

let it be given that $x^2+14x = 15$,

here if we try to throw the left-hand side into the form $(x+a)^2$, the auxiliary number a is given by

$$2a = 14, \quad \text{so} \quad a^2 = 49;$$

and the equation becomes

$$x^2+14x+49 = 15+49 = 64,$$

or $$(x+7)^2 = 8^2;$$

whence $$x = -7 \pm 8,$$

wherefore x = either 1 or -15.

One more plainly numerical example:

$$x^2-6x=20,$$

here a is manifestly 3, and the equation becomes

$$x^2-6x+9 = 29,$$

or $$(x-3)^2 = 29;$$

wherefore $$x=3 \pm \surd(29)$$

and it can only be carried further by extracting the numerical and incommensurable root.

Now a slightly more general touch:

given $x^2-12x = n$.

To reduce this to the form $(x-6)^2$, we must add 36 to each side, getting $x^2-12x+36 = n+36$

whence the solution is $(x-6) = \pm \surd(n+36)$

Finally let the given equation be

$$x^2 - 2ax = n^2.$$

Complete the square on the left hand side,

$$x^2 - 2ax + a^2 = n^2 + a^2$$

it becomes

$$(x - a)^2 = n^2 + a^2$$

or

$$x = a \pm \sqrt{(n^2 + a^2)};$$

which is essentially a general result.

The form of this result is easy to remember, and it is really general; for if the quadratic equation had been given in the manifestly general form

$$Ax^2 + Bx + C = 0,$$

where the coefficients A, B, C, stand for any known quantities of any kind whatever, it can be reduced to the above form by first dividing by A, and then instituting comparisons between it and the above: for we then see that correspondence requires the following identities:

$$n^2 = -\frac{C}{A} \quad \text{and} \quad 2a = -\frac{B}{A},$$

so that

$$a^2 = \frac{B^2}{4A^2};$$

wherefore the solution of the general quadratic is

$$x = -\frac{B}{2A} \pm \left(\frac{B^2}{4A^2} - \frac{C}{A}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2A}\{-B \pm \sqrt{(B^2 - 4AC)}\}.$$

But this should not be given to pupils for a long time yet, and perhaps we have already been attracted a little further than in the present book is legitimate. The pupil should by no means be thus hurried. A month's practice at the numerical and factor forms of expression may be desirable before passing to even slightly more general forms.

CHAPTER XV.

Extraction of Simple Roots.

THE last arithmetical lines of Chapter XIII. practically asserted that $18 = 5832^{\frac{1}{3}}$; and it can easily be verified by multiplication that

$$18 \times 18 \times 18 = 5832$$

or that 18 is the cube root of 5832.

Here then is a method, automatically suggested, for finding cube and other roots:—Analyse the number into factors whose roots are known, as 5832 was analysed into 27 and 216, at the end of the chapter referred to. It cannot always or often be done, but whenever it can it is quite the best way.

But to be able to apply this method we must cultivate an eye for factors, and we must also recognise or know by heart a certain collection of cube and square numbers.

Thus $$1728 = 12 \times 12 \times 12$$

or the cube root of 1728 is 12.

This is easy to remember because it represents the number of cubic inches in a cubic foot. In a country with a purely decimal system of measures, this fact would not be known with the same ease. They would know well however that

$$1000 = 10 \times 10 \times 10$$

or that the cube root of a thousand is ten; and so do we.

We may also know that 729 has 9 as its cube root, since it evidently equals 81×9, that is $9^2 \times 9$ or 9^3. The cube root of 343 is 7.

The square root of 10,000 is 100, and the square root of this is 10; but what its cube root is is not so easy to say.

The cube root of 1000 is 10; but what its square root is is not so easy to say.

The fifth root of 32 is 2, but neither its square root nor cube root is simple.

It is valuable to remember thoroughly that 2 is the cube root of 8.

The square root of a million is 1000,
the cube root of a million is 100;
the sixth root of a million is 10.

But there is no need to trouble about remembering any more than a few ordinarily occurring square roots and cube roots; for the sixth and higher roots are seldom wanted, and they can usually be derived from square and cube roots.

A number like 64 can resolve itself into 8×8
or into 4×16.

Its cube root is therefore easily stated as 2×2, viz. 4, and its square root as 2×4, viz. 8.

$$144 \text{ again} = 12 \times 12 \text{ and also} = 9 \times 16,$$

and either pair of factors gives its square root but not its cube root.

Surds.

Now let us proceed to ask what is the root of a number like 12 [where the word "root" is used alone, square root is understood]. We can resolve it into factors and find the root of each

$$12 = 4 \times 3,$$

so $$\sqrt{12} = \sqrt{4} \times \sqrt{3} = 2 \times \sqrt{3}.$$

So the result may be stated that

$$\sqrt{12} = 2\sqrt{3}.$$

Similarly $$\sqrt{8} = 2\sqrt{2}.$$

Again, let us find the cube root of say 24.

$$24 = 8 \times 3 \quad \text{and} \quad \sqrt[3]{8} = 2;$$

so $24^{\frac{1}{3}}$, which is often written $\sqrt[3]{24}$, $= 2\sqrt[3]{3}$.

Note the following:

$$\sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2},$$

$$\sqrt{32} = \sqrt{\left(\frac{64}{2}\right)} = \frac{\sqrt{64}}{\sqrt{2}} = \frac{8}{\sqrt{2}}.$$

Thus it would appear that $4\sqrt{2}$ must equal $\frac{8}{\sqrt{2}}$. If we multiply or divide each of these numbers by $\sqrt{2}$ we can easily verify this asserted quality. For multiplication by $\sqrt{2}$ makes them both 8; division by $\sqrt{2}$ makes them both 4.

Verify the following statements:

$$\sqrt[3]{56} = 2\sqrt[3]{7}.$$
$$\sqrt{20} = 2\sqrt{5}.$$
$$\sqrt{18} = 3\sqrt{2}.$$
$$\sqrt{27} = 3\sqrt{3}.$$
$$\sqrt{72} = 3\sqrt{8} = 6\sqrt{2}.$$
$$\sqrt{50} = 5\sqrt{2}.$$
$$\sqrt{(200)} = 10\sqrt{2}.$$
$$\sqrt{216} = 6\sqrt{6}.$$
$$\sqrt{(360)} = 6\sqrt{10}.$$
$$\sqrt{(810)} = 9\sqrt{10}.$$
$$\sqrt{(490)} = 7\sqrt{10}.$$
$$\sqrt{(125)} = 5\sqrt{5}.$$
$$\sqrt{(1000)} = 10\sqrt{10}.$$
$$\sqrt{(1728)} = 12\sqrt{12}.$$
$$\sqrt{343} = 7\sqrt{7}.$$
$$\sqrt{512} = 8\sqrt{8} = 16\sqrt{2}.$$

The last five are all **cube numbers**, and their value suggests a rule for expressing the square roots of any cube number; *e.g.*

$$\sqrt{27} = 3\sqrt{3}, \quad \text{or} \quad \sqrt{n^3} = n\sqrt{n}.$$

N.B.—The best way of interpreting the word "verify" at the head of the above set of examples, or in any similar place, is for the pupil to take the left-hand expression, and try as an exercise, independently, to simplify or otherwise express it, and see if he can reduce it to the form given on the right-hand side. He will thus perceive that numbers which have two factors can have the expression for their roots put into another form, which is often a more simple form; and that a large number of roots could be found numerically if the roots of a few prime numbers were known.

The number ten, as usual, has an unfortunate disability, in that neither of its factors is a perfect square, as one of the factors of 12 is. All we can do with $\sqrt{10}$ therefore is to say that it equals $\sqrt{2}\sqrt{5}$, which is of extremely little use. It is better kept as $\sqrt{10}$ and considered to be one of the things to be found. Since the sq. root of 9 is 3, while 4 is the sq. root of 16, the pupil may make a guess and try whether 3·1 approximates to the sq. root of 10 or not. He can easily do this by multiplying 3·1 by itself. By this means he can gradually correct its value. He can in a similar way make guesses also at $\sqrt{20}$ and $\sqrt{30}$ and $\sqrt{50}$. Let him try.

No simplification, by resolution into factors, can be made with any such numbers as

$$\sqrt{2},\ \sqrt{3},\ \sqrt{5},\ \sqrt{7},\ \sqrt{11},\ \sqrt{13},\ \sqrt{17},\ \sqrt{19},\ \sqrt{23},$$

and so on: that is, no simplification of this kind can be applied to any root of any prime number, naturally.

The roots of even numbers may always have a $\sqrt{2}$ exhibited:

$\sqrt{6}$ may be written $\sqrt{3}\sqrt{2}$,

$\sqrt{10}$,, ,, $\sqrt{5}\sqrt{2}$,

$\sqrt{14}$,, ,, $\sqrt{7}\sqrt{2}$,

but it is seldom useful to express them in this way.

Let us see if we must draw a distinction between $(\sqrt{9})^3$ and $\sqrt{(9^3)}$, that is between the cube of root nine and the root of nine cubed. Now

$$(\sqrt{9})^3 = 3^3 = 27,$$

while $$\sqrt{(9^3)} = \sqrt{(729)} = 9\sqrt{9} = 27 \text{ likewise.}$$

So they turn out to be the same.

The cube of a root appears to be equal to the root of a cube. That is curious, and may well be unexpected. It is not the sort of thing at all safe to **assume.** Plausible assumptions are always to be mistrusted and critically examined; occasionally, as in this instance, they turn out true.

Let us consider the fact more generally, and see whether it is always true that

$$\sqrt{(n^3)} = (\sqrt{n})^3.$$

The other and more expressive notation for roots will here come to our aid.

$$\sqrt{(n^3)} \text{ may be written } (n^3)^{\frac{1}{2}},$$

$$\text{and } (\sqrt{n})^3 \text{ may be written } (n^{\frac{1}{2}})^3,$$

so it looks as if both could be written as $n^{\frac{3}{2}}$ or $n^{1\frac{1}{2}}$, or, more properly, $n^{1+\frac{1}{2}}$.

This last is a thing we have not yet learnt how to interpret. We may assume however, as an experimental fact, that

$$\sqrt{(n^3)} = \sqrt{n^2}\sqrt{n} = n\sqrt{n},$$

hence the interpretation $n\sqrt{n}$, that is $n \times n^{\frac{1}{2}}$, suggests itself for $n^{1\frac{1}{2}}$ or $n^{1+\frac{1}{2}}$; and it is the right interpretation.

Here again (as on page 126) we have arrived at a striking circumstance about indices, which is now well worthy of examination.

CHAPTER XVI.

Further consideration of indices.

THERE are two things to which we might now appropriately turn our attention: one is the numerical calculation of all manner of roots, for instance, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{10}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$, etc., $\sqrt[5]{2}$, $\sqrt[5]{100}$, and so on; evidently a large subject, since we may require to find any root of any number; the other is the discussion of that curious property of indices, which has been dimly suggested by certain of the examples chosen, viz. the suggestion that

$$x^{n+m} = x^n \times x^m,$$

and that

$$(x^n)^m = (x^m)^n = x^{mn}.$$

Of these two directions along which we could now continue the discussion, the latter is undoubtedly the easier, and so we will proceed this way first; and incidentally we shall find ourselves led to a very practical and grown-up way of dealing with the former more difficult line of advance.

What we found experimentally (on page 126) was that

$$3 \times 3^2 = 27 = 3^3;$$

also that

$$3^2 \times 3^3 = 9 \times 27 = 243 = 3^5.$$

And so we might have taken other instances:

$$2^2 \times 2^4 = 4 \times 16 = 64 = 2^6,$$

$$2 \times 2^5 = 2 \times 32 = 2^6,$$

$$2^3 \times 2^3 = 8 \times 8 = 2^6,$$

$$2 \times 2^2 \times 2^3 = 2 \times 4 \times 8 = 2^6.$$

What does all this look like?

Manifestly it looks as if to effect a product among the powers of a given number, we must add the indices of the several powers. It looks like

$$2^3 \times 2^4 = 2^7,$$
$$2^3 \times 2^8 = 2^{11},$$
$$6^3 \times 6^2 = 6^5,$$
$$2^{\frac{1}{2}} \times 2^2 = 2^{\frac{5}{2}} = (\sqrt{2})^5 = \sqrt{(2^5)} = \sqrt{32} = 4\sqrt{2},$$
$$2^4 \times 2^{\frac{1}{2}} = 2^{4\frac{1}{2}} = 2^{\frac{9}{2}} = \sqrt{(2^9)} = (\sqrt{2})^9,$$
$$= \sqrt{(512)} = \sqrt{(2 \times 256)} = 16\sqrt{2}.$$

Now when the indices are whole numbers it is very easy to see the reason of this simple rule. What does 2^4 mean? It means that four factors each of them 2 are to be multiplied together. The index is only an indication of how many times the similar multiplication is to be performed.

2^4 means simply $2 \times 2 \times 2 \times 2$; the number of multiplication signs being one less than the index, *i.e.* one less than the number of factors of course. Similarly 2^3 is merely an abbreviation for $2 \times 2 \times 2$. Hence

$$2^4 \times 2^3 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2,$$

that is seven 2's are to be multiplied together; and so it is naturally indicated by 2^7.

The index counts the number of similar factors; hence when the factors are increased in number the index shows the simple increase; but the effect of the continued multiplication on the resulting number may be prodigious.

The anecdote about the nails in the horse's shoes here appropriately comes in:

A man, who objected to the price asked for a horse, was offered the horse as a free gift, thrown into the bargain, if he would buy merely the nails in its shoes, of which there were 6 in each foot; at the price of a farthing for the first nail, 2 farthings for the second, 4 farthings for the third, 8 for the fourth, and so on. The offer being accepted, he had to pay £17,476 5s. $3\frac{3}{4}$d. for the nails; and he did not consider the horse cheap.

The number of farthings in this sum is very great, but it is simply one less than 2^{24}.

If a beginner wishes to verify the above by multiplying 2 by itself 23 times, he can easily do it, though it will take a little time; and he can then reduce the result to pounds shillings and pence, as he has been no doubt so well taught how to do. It is not a grown-up way of ascertaining 2^{24}, but it serves. (Reference to pp. 166 and 259 may be convenient.)

If he is properly sceptical about the magnitude and correctness of the above sum, he should do it. It is good practice in easy multiplication; and sums which are set by the pupil to himself are likely to secure greater attention from him than those enforced from outside. It is probably desirable that children should often **set** sums as well as work at them. I would even sometimes encourage them to set examination papers. It is a good way of getting behind the scenes.

As regards the verification of $a^m \times a^n = a^{m+n}$ therefore, the idea is very simple, so long as m and n are whole numbers; because it is a mere matter of counting the number of similar factors.

When we say that five sixes multiplied together equal 7776, we are employing the number five in this very way. The expression 6^5 does not mean five sixes added together, or 30; but it means five sixes multiplied together, yielding a much larger result.

So also six tens multiplied together make a million, whereas added they only make 60. In fact, as we said before, page 56 and Chap. XII., while multiplication is abbreviated addition, involution is abbreviated multiplication.

Fractional indices.

When the indices m and n are fractions, the idea they express is not so simple, and the above relation $a^m \times a^n = a^{m+n}$ is not so easily justified; but we may be willing to accept it by analogy and see how it works.

If asked wherein the proof consists for fractional indices, we must answer in "consistency," constant coherence and agreement with results so obtained, and in corresponding convenience of manipulation.

At one time $2^{\frac{1}{2}}$ and such like were called irrational quantities because it was difficult to attach a commonsense significance to "2 multiplied by itself half a time"; and it is certainly not to be interpreted as half 2 multiplied by itself, for that would be unity.

There is nothing irrational about this quantity however: it has a value approximately 1·4142 ... though it will hereafter be found that it will not express itself exactly by a finite series of digits in any system of notation whatever.

It may rightly be styled "incommensurable" therefore, but it is in no sense irrational.

"Irrational" however was a term at one time applied to any power of a number whose index was not a positive integer.

The thing has to be mentioned, for historical reasons, but the term "irrational" should now cease to be used. The term "**surd**," being meaningless, may be employed if we like, but it is never really wanted: it only serves as a heading to a chapter to indicate its contents.

Negative indices.

But let us go on and ask how shall we interpret the expression if one of the indices be not fractional but negative?

For instance, how shall we interpret

$$3^{-2}, \text{ or } 2^{-6}.$$

Suppose for instance we had

$$a^m \times a^{-n},$$

we should naturally say that the result must be a^{m-n}.

Very well, let $m = 2$ and $n = 3$,

then $$a^2 \times a^{-3} = a^{-1}.$$

What does a^{-1} mean?

How can we multiply a number by itself a negative number of times? At first the term "**irrational**" was applied to such quantities as these: but a consistent interpretation was soon found for them. If addition of indices means multiplication, it is natural that subtraction of indices shall mean division.

Make the hypothesis therefore that a^{m-n} can be interpreted as $a^m \div a^n$, and let us see how that works.

Suppose we had $2^5 \div 2^3$, it could be written out in full,

$$\frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2},$$

and the result after cancelling would be 2×2,

that is $$2^2 = 2^{5-3}.$$

The whole thing is therefore quite simple.

Take other examples:

$$3^{(6-5)} = \frac{3^6}{3^5} = 3,$$

$$a^{7-4} = \frac{a^7}{a^4} = a^3,$$

$$x^{a-b} = \frac{x^a}{x^b},$$

$$x^{a-b+c} = \frac{x^{a+c}}{x^b} = \frac{x^a x^c}{x^b} = x^a \times x^c \div x^b,$$

$$7^{4-2+1} = \frac{7 \times 7^4}{7^2} = 7^3 = \mathbf{343},$$

$$7^{a-2} = \frac{7^a}{7^2} = \frac{1}{49}7^a,$$

$$7^{a-a} = \frac{7^a}{7^a} = 1.$$

This last is a most interesting and useful result.

If the index is zero, the quantity, whatever it may be, is reduced to unity; for

$$a^{m-m} = \frac{a^m}{a^m} = 1;$$

it equals 1 whatever a may be.

$a^0 = 1$ is the brief summary of this important consequence of our notation. The index 0 would have been hard to interpret, just as fractional and negative indices were hard to interpret, but fortunately it thus interprets itself.

A **negative** sign applied to an **index** turns out therefore to have the effect of giving the **reciprocal** of the quantity; for since

$$x^{m-n} = \frac{x^m}{x^n},$$

we have only to take the case where m is zero, in order to get

$$x^{-n} = \frac{x^0}{x^n} = \frac{1}{x^n}.$$

Hence

$$2^{-1} = \frac{1}{2},$$

$$2^{-2} = \frac{1}{4},$$

$$3^{-2} = \frac{1}{9},$$

$$3^{-3} = \frac{1}{27},$$

$$2^{-5} = \frac{1}{32}.$$

Hence while $2^{\frac{1}{2}}$ means $\sqrt{2}$, we thus find that 2^{-1} means $\frac{1}{2}$; or, in general,

$$x^{-n} = \frac{1}{x^n} \text{ and } x^{-1} = \frac{1}{x}.$$

Take the last simple and useful mode of expression. To verify it, simply multiply both sides by x, thus

$$x^{-1} \times x^{+1} = x^{-1+1} = x^0 = \frac{x}{x} = 1$$

Similarly $$x^{-2} = \frac{1}{x^2} = \left(\frac{1}{x}\right)^2 = (x^{-1})^2,$$

$$a^{-n} = \frac{1}{a^n} = (a^n)^{-1} = (a^{-1})^n.$$

And this suggests powers of powers; like $(10^3)^2$, that is the square of a thousand, which is a million, 1 followed by 6 ciphers, or 10^6.

So also $(10^6)^2 = 10^{12}$ = a billion; the indices being in this case multiplied to give the result.

So now we leave addition and subtraction among indices, which merely meant multiplication and division among the quantities themselves, and begin to study multiplication among indices.

Consider for instance what the meaning should be of $4^{3\times 2}$;

$$\text{it equals } 4^6 = (4^3)^2 = (4^2)^3 = 4096,$$

$$x^{mn} = (x^m)^n = (x^n)^m.$$

So **multiplication among indices means involution** among the quantities themselves.

So also **division among indices will signify evolution** among quantities, thus

$$7^{\frac{3}{2}} = (7^3)^{\frac{1}{2}} = (7^{\frac{1}{2}})^3,$$

$$x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m} = (x^{\frac{1}{n}})^m = (\sqrt[n]{x})^m,$$

the order of the factors $\left(\text{which in this case are } m \text{ and } \frac{1}{n}\right)$ being indifferent.

If it were worth while we might proceed further, and consider what would be the meaning of the process "involution" applied to indices; how would that affect the quantities themselves? What for instance is the meaning of 2^{2^2}? but it is a mere curiosity and is hardly worth while. Suffice it to say that the numbers so reached become rapidly prodigious. 10^{10} is a number with ten ciphers after the 1, or ten thousand million; but $10^{10^{10}}$ possesses a hundred cyphers, and represents a number far greater than that of all the atoms of matter in the whole solar system—earth, sun, and all the planets,—notwithstanding the fact that a speck containing a million-million atoms is only visible in a high power microscope.

CHAPTER XVII.

Introduction to Logarithms.

THE equation $y = x^n$, that is, the n^{th} power of x, may be equally expressed as $x = y^{\frac{1}{n}}$, that is, the n^{th} root of y; this is not an inverse expression, but the same in inverse form.

So also the equation $xy = 1$, which represents a multiplication sum, can also be written $y = 1/x$, which represents a division sum; and $x^2y^2 = c^2$ can appear as $c = \pm xy$, the double sign representing an ambiguity or double solution, because either $+c$ or $-c$ would when squared give the right result.

If y is the n^{th} power of x, it is easy to say that x is the n^{th} root of y; we can also say that n is the index or exponent of x which yields the value y; but how are we to express the relation that n bears to y?

It is a thing we have not yet come across.

It is called a **logarithm**; it involves a reference to both x and y; it is called the logarithm of y to the base x.

Let us understand this matter.

Write down $$100 = 10^2,$$

2 is called the logarithm of a hundred to the base ten.

Conversely $10 = 100^{\frac{1}{2}}$, so $\frac{1}{2}$ might be called the logarithm of ten to the base a hundred.

Write down $$25 = 5^2,$$

2 is the logarithm of 25 to the base 5.

The logarithm of a number is defined as the index of the power to which the base must be raised in order to equal the given number.

Thus if we are told that 3 is the logarithm of a thousand to the base ten, it is another mode of stating that $10^3 = 1000$.

So 3 is the logarithm of 8 to the base 2,
2 is the logarithm of 49 to the base 7,
5 is the logarithm of 32 to the base 2,
3 is the logarithm of 216 to the base 6,
4 is the logarithm of 81 to the base 3,

and so on.

It looks a cumbrous and roundabout mode of expressing what is more neatly expressed by the index notation, but it is an exceedingly practical and convenient mode of statement all the same, and is a great help in practical computation.

What is the logarithm of 343 to the base 7? Answer, 3.

What is the logarithm of a million to the base 10? Answer, 6.

What is the logarithm of 64? It is 6 to the base 2, and 2 to the base 8.

What is it to the base 10? Answer, something less than 2 and more than 1.

What is the logarithm of 10 to the base 10, or of any number to its own base? Answer, unity, for $a = a^1$.

What is the logarithm of unity itself?

The answer is 0, to any base, because $1 = a^\circ$.

What is the logarithm of a fraction, say $\frac{1}{4}$, to the base 2? Answer, a negative quantity, in this instance -2, because $\frac{1}{4} = 2^{-2}$.

So also -2 is the logarithm of $\frac{1}{100}$ to the base 10, because $\frac{1}{100} = 10^{-2}$. And the fact that $\frac{1}{\text{million}} = 10^{-6}$ can be expressed by saying that -6 is the log of a millionth to the base 10.

It appears therefore that the logarithms of reciprocals or of numbers less than 1 are negative, the log of 1 itself being 0.

This is satisfactory. Everything greater than 1 has a positive logarithm, everything less than 1 has a negative logarithm; provided always that the base itself is greater than one. The further a number is removed from 1 both ways, whether in the direction of greatness or of smallness, the larger numerically is the logarithm; but it is positive bigness in the one case, negative bigness in the other. It is natural therefore that the logarithm of 1, to any base, should be zero.

Mathematicians know how to calculate the log of any number, no matter how complicated, and they have recorded the results in a book called a table of logarithms; just as grammarians and scholars know how to translate any foreign word, and have recorded the results in books called dictionaries. A Table of Logarithms is to be used like a dictionary. It can be readily used, and is used every day, by those who would find it difficult to construct it. It should puzzle children sometimes how the meaning of words in dead foreign languages were ascertained; they mostly take it for granted and do not think about it. So also, for a time, and until they make some approach to becoming budding mathematicians, they need not learn how to compute a table of logarithms; but they must imbibe a clear idea as to their meaning. They must also, and that is an easier matter still, learn their practical use, and be able to use a table as they have learnt how to use a dictionary.

CHAPTER XVIII.

Logarithms.

WHEN we express a number thus:

$$64 = 8^2,$$
$$1000 = 10^3,$$
$$32 = 2^5,$$

or, in general, $$n = a^x,$$

we are said to express it "exponentially," that is, by means of the index or "exponent" of the power to which a certain other number called a **base** is to be raised in order to be equal to the given number.

In the above equation n stands for the number, a for the base, and x for the index or exponent of that base.

The question naturally arises, what relation does x bear to n, for it manifestly depends upon both n and a? If the base has been specified and kept constant, then x will vary only as n varies. It is plain that x will increase as n increases, but not nearly so fast.

Take a few examples, and first take the number 2 as base:

$$2 = 2^1,$$
$$4 = 2^2,$$
$$8 = 2^3,$$
$$16 = 2^4,$$
$$32 = 2^5$$
$$64 = 2^6,$$
$$\ldots \quad \ldots$$

$$1024 = 2^{10},$$
$$\ldots \quad \ldots$$
$$16,777,216 = 2^{24}.$$

Here the **index** runs up slowly, 1, 2, 3, 4, etc., according to what are called the "natural numbers"; whereas the **number** on the left-hand side runs up very quickly. The **index** is said to progress "arithmetically," that is, by equal additions; the **number** on the other hand is said to progress "geometrically" (a curious use of the word), that is, by equal multiplications. There is evidently some law connecting the index and the number, when a base is given; and the following nomenclature is adopted:

5 is called the logarithm of 32 to the base 2;
3 is the logarithm of 8 to the base 2;
4 is the log of 16 to base 2;
6 = log 64 (base 2),

which is usually abbreviated still further:

$$6 = \log_2 64;$$
$$10 = \log_2 1024,$$

the base being indicated as a small suffix to the word log.

Make now a more complete table; first of powers:

$$1 = 2^{\circ},$$

$2 = 2^1,$	- - -	$\frac{1}{2} = 2^{-1},$
$4 = 2^2,$	- -	$\frac{1}{4} = 2^{-2},$
$8 = 2^3,$	- - -	$\frac{1}{8} = 2^{-3},$
$16 = 2^4,$	- - -	$\frac{1}{16} = 2^{-4},$
$32 = 2^5,$	- - -	$\frac{1}{32} = 2^{-5},$
$\ldots \quad \ldots$		$\ldots \quad \ldots$
$1024 = 2^{10},$	- -	$\frac{1}{1024} = 2^{-10},$
$\ldots \quad \ldots$		$\ldots \quad \ldots$

and then of the corresponding logarithms:

From the above table it follows that (with the base 2)

$$\log 1 = 0,$$

$\log 2 = 1,$	$\log \frac{1}{2} = -1,$
$\log 4 = 2,$	$\log \frac{1}{4} = -2,$
$\log 8 = 3,$	$\log \frac{1}{8} = -3,$
$\log 16 = 4,$	$\log \frac{1}{16} = -4,$
$\log 32 = 5,$	$\log \frac{1}{32} = -5.$
... ...	... ...

It would be a good thing to plot both these tables on squared paper, representing for the first the indices 1, 2, 3, 4 as horizontal distances, and the numbers 2, 4, 8, 16 as vertical distances; and for the second measuring distances to represent the 2, 4, 8, 16 numbers horizontally, and the logarithm numbers 1, 2, 3, 4 vertically.

The first is called an exponential curve, or curve of exponents or indices; the second is called a logarithmic curve, or curve of logarithms. The two curves turn out to be identically the same, only differently regarded,—to make their identity apparent, the paper can be turned round and looked through at the light.

If drawn on the same sort of squared paper the curves will fit. They may either of them be said to represent the relation between Geometrical and Arithmetical progression: in one direction distances proceed arithmetically, or by equal differences; in the other geometrically, or by equal factors.

These curves will do for any base, if their *scale* is suitably interpreted. The divisions we have labelled 2, 4, 8, etc., may equally well be considered to represent 3, 9, 27, etc., or a, a^2, a^3, etc., or 10, 100, 1000, etc.

That is the advantage of a curve. Once drawn, it represents to the eye a *general* kind of relationship; and nothing but an interpretation of its scale is necessary to make it fit any required instance of that relationship. The shape of the above suggested curve is drawn on pages 101 and 179.

Verify the following statements:

$$
\begin{aligned}
3 &= \log 27 \text{ to the base } 3, \\
4 &= \log_3 81, \\
6 &= \log_3 729, \\
2 &= \log_4 16, \\
4 &= \log_4 256, \\
5 &= \log_4 1024, \\
2 &= \log_6 36, \\
3 &= \log_6 216, \\
3 &= \log_7 343, \\
3 &= \log_9 729, \\
2 &= \log_9 81, \\
\text{but } 9 &\neq \log_2 81, \\
2 &= \log_{12} 144, \\
3 &= \log_{12} 1728.
\end{aligned}
$$

This last seems a curious and roundabout way of expressing the fact that $12 \times 12 \times 12 = 1728$, and if it did not turn out practically very convenient there would be no justification for introducing such a complication as the logarithmic notation instead of the index notation; but it is constantly to be noticed, when a new notation has been introduced into mathematics, that it confers on us an extraordinary power of progress, and enables difficulties further on to be dealt with which were before intractable.

Any complication which is of no use—or let us say of no obvious and well-known use—anything which should not be familiar to every educated person—is not treated of in this book; the justification of any notation is that though for the expression of simple and already well-known facts it may look cumbrous and inexpressive, yet when we want to express harder and at present unknown ideas it becomes helpful and luminous.

Common practical base.

The case when the logarithmic base is the same as the base adopted for our system of numerical notation, is worth special attention, because it is the one most frequently used in practice. What the base for numerical notation may be, is, as we know, a pure convention; and, as we have explained, it is perhaps an unfortunate but now irremediable convention that the base of notation is **ten.** It does not follow that the logarithmic base must also be ten: it is perhaps possible to find a **natural** base, involving no convention. If so, such a base would of course be important and interesting; but meanwhile we will take **ten** as the base also of a practical system of logarithms.

Let us first make a table of powers of ten.

$$1 = 10^0,$$

$$\begin{array}{rclcrcl} 10 &=& 10^1, & \quad & \frac{1}{10} &=& 10^{-1}, \\ 100 &=& 10^2, & & \frac{1}{100} &=& 10^{-2}, \\ 1000 &=& 10^3, & & \frac{1}{1000} &=& 10^{-3}, \\ \cdots && \cdots & & \cdots && \cdots \\ 1{,}000{,}000 &=& 10^6, & & \frac{1}{1{,}000{,}000} &=& 10^{-6}, \\ \cdots && \cdots & & \cdots && \cdots \end{array}$$

Whence it follows that (with base 10)

$$\log 1 = 0,$$

$$\begin{array}{rclcrclcl} \log 10 &=& 1, & \quad & \log \frac{1}{10} &=& \log \cdot 1 &=& -1, \\ \log 100 &=& 2, & & \log \frac{1}{100} &=& \log \cdot 01 &=& -2, \\ \log 1000 &=& 3, & & \log \frac{1}{1000} &=& \log \cdot 001 &=& -3, \\ \cdots && \cdots & & && \cdots && \cdots \\ \log 1{,}000{,}000 &=& 6, & & \log \frac{1}{1{,}000{,}000} &=& \log \cdot 000001 &=& -6, \\ \cdots && \cdots & & && \cdots && \cdots \end{array}$$

Hence (with base 10) the logarithms of numbers between 10 and 100 lie between 1 and 2, that is to say consist of 1 and a fraction: the log of 11 will be 1 and a small fraction, the log of 99 will be 1 and a large fraction—very near to 2 in fact. Consequently with all double-digit numbers the characteristic property of the logarithm is that it begins with 1.

All numbers which consist of three figures lie between 100 and 1000, and these have the characteristic 2; that is to say they all consist of 2 + a fraction. This is true even of such a number as 999·99, provided the 9's are not repeated for ever; because although the log of such a number is very nearly 3, it is not quite 3 until 1000 is reached.

A number consisting of five digits will have a log whose characteristic is 4, and so on; the characteristic is always equal to the number of digits on the left of the unit digit, which is taken as a zero of reckoning. Thus the characteristic of the log of any of the following numbers (1200, 1728, 5760, 9898, 1431·8, 1696·25) is 3.

The logarithm of every fraction between 0 and ·1 will be a negative fraction: it will not be quite equal to − 1, but it may be put equal to − 1 plus a positive fraction.

The logarithm of every number between ·1 and ·01 will lie between − 1 and − 2, and therefore may be expressed either as − 1 minus a fraction, or as − 2 plus a fraction; and the latter is the usual plan.

The rule for the characteristic therefore is to count always to the first, *i.e.* the most important significant figure, starting from the units place as zero. On this method of expression it is easy to write down the characteristic of the logarithm of any number at sight.

The best plan is to employ the term "order," connoting by the *order* of the number the index of the power to which the base say ten must be raised in order to give a number with that number of digits. *E.g.* the order of 100 is 2, because it equals 10^2, and all the numbers 121, 256, 780, 900 may be technically designated as of the same "order"; because, though greater than 10^2, they are less than 10^3; and the amount by which they exceed 10^2 is shown by the fractional part of the logarithm, not by its integer part or characteristic. But 1000 is of the order 3, and so likewise is 1728, etc.

17 is of the order 1, and so is 14·58,
4 is of the order 0, and so is 4·6,
·3 is of the order − 1, and so are ·35 and ·78,
·02 and ·035 and ·016 are of the order − 2.

Accepting this nomenclature, which is useful in quite rudimentary arithmetic, *e.g.* in long division and the like, we are able to say simply that the characteristic of a logarithm is the "order" of its number.

Let there be no confusion between the table on page 169 and the one on page 167 to base 2. They involve different bases; and though the base is not expressed every time, but only in the heading, that is merely because of the needless trouble of frequently printing or writing suffixes, like this:

$$3 = \log_2 8 = \log_6 216 = \log_{10} 1000.$$

Examples.

The characteristic of the logarithm of the following numbers (also called the "order" of the number itself) is as here given:

The logarithm, to base 10, of each of the numbers

5, 8·7, 1·23, 9·99, 1·111 has the characteristic 0;

and this is the "order" of each number.

Of each of the numbers

	17,	94,	17·65,	11·1	the order is	1
of	300,	981·4,	101·01,		it is	2
,,	·17,	·94,	·11101,		it is	− 1
,,	·08,	·01,	·0999,		it is	− 2
,,	·002,	·0056,	·009846,		it is	− 3

Examples for Practice.

Write down the characteristic of the logarithm of each of the following numbers, (in other words express the "order" of each number):

56,	108,	56·75,	108·001,	17·9909,
8·3,	8300,	5065·2,	5·0652,	8×10^6,
·56,	·0056,	·008309,	1·0056,	10·001,
99·9,	9·9,	·099,	256000,	·0000256.

Fundamental relations.

There are a few fundamental properties appropriate to logarithms belonging to any base whatever.

One of them is that $\log 1 = 0$,
and another is that $\log(\text{base}) = 1$,
but there are others which we have already several times hinted at.

Let us recollect once more what an index or exponent signifies. It signifies the number of similar factors which have to be multiplied together. For instance,

$$2^5 = 2 \times 2 \times 2 \times 2 \times 2,$$

$$6^4 = 6 \times 6 \times 6 \times 6,$$

or, in general,

$$a^3 = a \times a \times a,$$

$$a^n = a \times a \times a \times \ldots \text{ to } n \text{ factors.}$$

So now if we write a number N equal to any of these, as for instance $N = a^3$,
the index, or exponent, which we shall now call the logarithm of N to the base a, simply counts the number of times the base occurs as a factor in the number N.

$$N = a \times a \times a.$$

Suppose now we took some number not quite so easy to deal with as those in the examples we have hitherto considered, a number which cannot be represented as any simple power of any integer, say for instance the number 30; and ask what will the logarithm of 30 be to the base 10.

First of all we see that it must be between 1 and 2, because 1 is the logarithm of 10, while 2 is the logarithm of 100; and the logarithm increases with the number, but arithmetically instead of geometrically. So as 30 lies roughly half way geometrically between 10 and 100, it may be expected that its logarithm will be somewhere about halfway arithmetically between 1 and 2. It will be 1 + a fraction; and what that fraction is can be approximated to more or less closely by examining and measuring the logarithmic curve which we ought to have carefully drawn, as indicated on p. 179, and specially labelled so as to suit the base 10. Measuring that curve for the logarithm of 30, it suggests a value something like $1\frac{1}{2}$ or 1·5. This would be the result to "two significant figures," but if the curve has been carefully drawn, it might give us 3-figure accuracy, that is, would enable us to express the result correctly to 3 significant figures; in that case we might estimate log 30 to be about 1·48.

The number which really lies geometrically half-way between 10 and 100 would be $\sqrt{(1000)}$, since $10 : \sqrt{1000} = \sqrt{1000} : 100$; and $\sqrt{(1000)}$ is accordingly called the geometrical mean of ten and 100. Hence the logarithm of $\sqrt{1000}$ is exactly 1·5 or $1\frac{1}{2}$. Similarly the logarithm of $\sqrt{10}$ is ·5 or $\frac{1}{2}$. It is a curious thing that though we do not yet know how to calculate the root of 10, we know its logarithm; and this suggests—what frequently happens—that the logarithm of the result of an arithmetical operation is easier to perceive than the result itself.

We can examine the curve again, to see if it will show us what number has a logarithm exactly 1·5; we shall see that it indicates something like 3·1 or 3·2, and if it were drawn carefully it might indicate 3·16. This is one of the values that we ought already to have arrived at by trial and error, as recommended on p. 152; taking different numbers between 3·1 and 3·2 and squaring them, to see how nearly the square would approach 10.

No number that we can select will, when squared, exactly equal ten. It has no square root that can be expressed numerically with exactness. Nor has any number except the square numbers, 1, 4, 9, 16, 25, etc. These **numbers**, *i.e.* the group ordinarily denoted by these symbols, are square numbers in any system of notation, their square roots can be numerically expressed precisely;* and for no other numbers can the same be done, however many fractions or combinations of fractions, or however many decimal places, are employed. Nor can it be done in any other system of notation. In other words,

$$\sqrt{2},\ \sqrt{3},\ \sqrt{5},\ \sqrt{7},\ \sqrt{1000},\ \text{etc., etc.,}$$

are all incommensurable.

But their logarithms are easily expressed, to any base whatever, in terms of the logarithm of the number itself to the same base. Thus

$$\log\sqrt{2} = \tfrac{1}{2}\log 2,$$
$$\log\sqrt{3} = \tfrac{1}{2}\log 3,$$
$$\log\sqrt{5} = \tfrac{1}{2}\log 5,$$
$$\log\sqrt{10} = \tfrac{1}{2}\log 10.$$

* *Caution.*—It is not intended, and it is not true, that the above *digits* express square numbers when interpreted in accordance with any scale of notation; for instance, the amount of money represented by 2/5 is not a square number of pennies, but the number we are accustomed to designate by 25 is a square number, and 25 coins can be easily arranged to form a square.

Hence to the base 10,

$$\begin{aligned}\log\sqrt{10} &= \tfrac{1}{2}\log 10 = 0{\cdot}5,\\ \log\sqrt{100} &= \tfrac{1}{2}\log 100 = 1{\cdot}0,\\ \log\sqrt{1000} &= \tfrac{1}{2}\log 1000 = 1{\cdot}5,\\ \log\sqrt{10000} &= \tfrac{1}{2}\log 10000 = 2{\cdot}0,\\ \log\sqrt{100000} &= \tfrac{1}{2}\log 100000 = 2{\cdot}5.\end{aligned}$$

Similarly we may guess that

$$\log\sqrt[3]{10} = \tfrac{1}{3}\log 10 = {\cdot}3333\ldots,$$

and so we may refer to the curve and see what number has the logarithm $\frac{1}{3}$, for that will be the cube root of ten. We find that it is about 2·1, or more nearly 2·14; and if we multiply this by itself three times, $2{\cdot}14 \times 2{\cdot}14 \times 2{\cdot}14$, we shall get a number not far off ten—a trifle greater than ten.

Similarly 21·4 will be approximately the cube root of ten thousand, and 214 of ten million.

No exact numerical specification of the cube roots of any number can be given, except of the cube numbers, that is, those numbers which, given in the form say of marbles, can be built up to represent cubes; namely such numbers as

1, 8, 27, 64, 125, and so on.

For the cube root of any other number, if it could be expressed, would be a fraction; and a fraction multiplied by itself necessarily remains a fraction; it can never yield an integer. **You cannot fractionate a fraction into a whole.**

This remark is further developed in Chapter XX.

We now know how to find the logarithm, to the base ten, of any power of ten, whether integral, negative, or fractional. Examples:

$$\log 10^3 = 3, \quad \log 10^{-3} = -3, \quad \log 10^{\frac{1}{3}} = \tfrac{1}{3};$$

$$\log 10^4 = 4, \quad \log 10^{-4} = -4, \quad \log 10^{\frac{1}{4}} = \tfrac{1}{4};$$

so generally $\log_{10} 10^x = x,$

and this may be easily generalised so as to apply to any base.

For $$\log a^x = x \log a,$$
but we know that $$\log(\text{base}) = 1,$$
so we see that the logarithm of any power of the base is equal to the index or exponent of the power, or

$$\log(\text{base})^x = x.$$

We have thus arrived at the original definition of a logarithm from which we started,—having reasoned "in a circle."

The advantage of reasoning in a circle is that we thereby check and verify to some extent the intermediate steps, for if any of them had been inconsistent we could not have worked round to our starting point; unless indeed we had happened to make a pair of errors which cancelled each other: a thing which is sometimes done—especially when the conclusion is consciously in our minds. Working round a circle of reasoning is in that case no adequate check. It is not possible to get round by any odd number of errors, but with an even number of errors it is possible though not very probable; unless indeed we know our destination too well beforehand.

The real test of truth is that it shall turn out to be consistent with everything else which we know to be true. No one chain of reasoning, however apparently cogent, is to be absolutely trusted,—for there is always the danger of oversight due to defective knowledge. Complete consistency is the ultimate test of truth; and convergence of a number of definite lines of reasoning is an admirable practical test.

CHAPTER XIX.

Further details about logarithms.

INVOLUTION and evolution become easy directly we employ logarithms:

To obtain any root, say the r^{th} root, of any number:

Find the logarithm of the given number to any base, calculate $\frac{1}{r}$th of this logarithm, then find the number which has this value for its logarithm to the same base; that number is the r^{th} root of the given number.

Or put it thus: utilising the logarithmic curve, page 179.

Take a length on the horizontal line as representing the given number; find its logarithm, as the vertical distance to the curve at this point; calculate $\frac{1}{r}$th of this length, and find on the curve a point whose vertical height is equal to it; then the foot of the perpendicular from this point marks out on the horizontal line a length which represents the rth root of the given number, on the same scale as the number itself was represented. Thus, for instance, $\frac{1}{3}$rd of the height of the curve at division 8, projected back horizontally, should meet the curve above the division 2; because 2 is the cube root of 8.

You see it is worth while to draw the curve neatly and carefully, so that fairly correct measurements may be made upon it. Besides, accurate drawing is a useful art, and it takes a little time to employ drawing instruments accurately so as to make no blots or smudges, and to get all lines uniformly thick and accurately passing through the points intended. It is an art worthy of cultivation for future use. Much information can be gained from such curves, not only in science but even in business and in politics.

It may be said that by this process of drawing and measuring, a logarithm or a root can after all only be attained approximately. Yes, but the same is true of any process, so far as accurate expression is concerned. A logarithm or a root in general requires an infinite series of digits to express it; all finite expression is approximate.

I do not however say that a mathematician would calculate logarithms or roots by such a curve: he would know plenty of other contrivances for such things, and perhaps we may know some of them later on; but he would not despise the curve method, at least in more really difficult investigations. He would use it frequently. But for mere logarithms he would use a table, somewhat as indicated in Chapters XXIX. and XXX.

Now let us see if we can calculate a few other logarithms. We can obtain any we want from the curve, but if we could obtain a few once for all, and label them, and then be able to express the logarithms of other numbers in terms of these, it might save us time and trouble; and besides it is a desirable and useful thing to be able to do.

We have managed to find the logarithm of any power of ten (p. 175), let us see if we can manage the logarithm of any product containing ten as one of its factors.

We have indeed already tried one of them, viz. 30 (see p. 173); let us try another, say 50, likewise to the base 10.

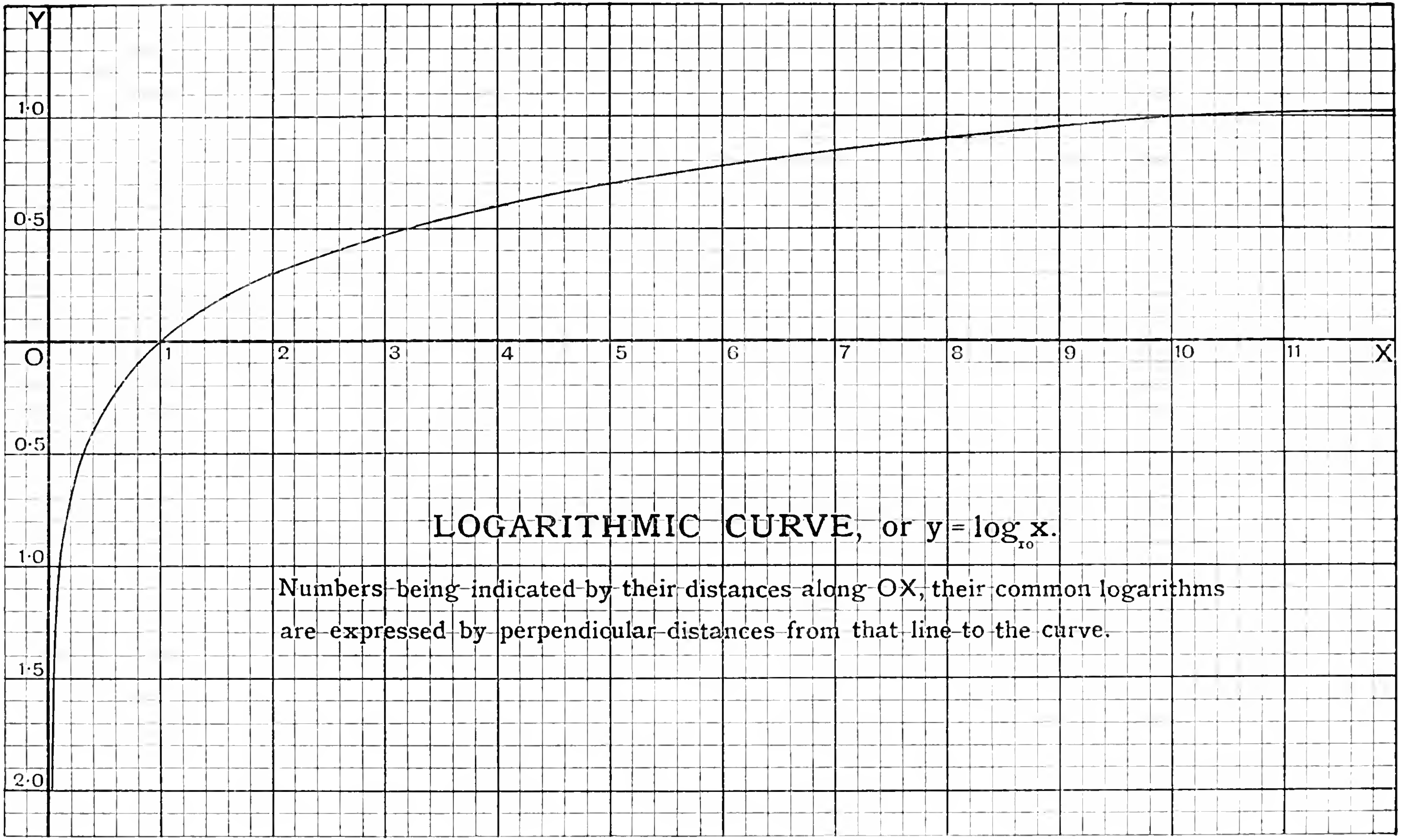
We want to find to what power 10 must be raised in order to equal 50.

Let x be the power, then

$$50 = 10^x$$

is an equation from which we have to find x.

This is only another mode of stating that $x = \log_{10} 50$.



Y
1·0
0·5
O
0·5
1·0
1·5
2·0
1
2
3
4
5
6
7
8
9
10
11
X
LOGARITHMIC CURVE, or $y = \log_{10} x$.
Numbers being indicated by their distances along OX, their common logarithms are expressed by perpendicular distances from that line to the curve.

But now resolve 50 into two factors, and write

$$5 \times 10 = 10^x,$$

then $$5 = \frac{10^x}{10} = 10^{x-1};$$

hence $$x - 1 = \log 5,$$

or $$x = 1 + \log 5 = 1{\cdot}7 \text{ about (by the curve).}$$

Thus $$\log 50 = \log 5 + \log 10,$$

which is a special case of a general assertion that

$$\log nm = \log n + \log m.$$

Examine this:

$$\text{Let } n = a^x, \text{ so that } x = \log n,$$

$$\text{and let } m = a^y, \text{ so that } y = \log m;$$

$$\text{then } nm = a^x a^y = a^{x+y};$$

$$\therefore \log nm = x + y = \log n + \log m.$$

Hence by using logarithms, multiplication is turned back into addition, just as involution was turned back into multiplication. So also division is turned into subtraction, just as evolution was turned into division.

The fundamental relations are as follows; and although we have stated them several times before, they are supremely important and will bear repetition.

Let $$a^x = n \text{ and } a^y = m,$$

so that $$x = \log n \text{ and } y = \log m,$$

then $$nm = a^x a^y = a^{x+y};$$

$$\therefore \log nm = x + y = \log n + \log m.$$

Furthermore, $$\frac{n}{m} = \frac{a^x}{a^y} = a^{x-y};$$

$$\therefore \log \frac{n}{m} = x - y = \log n - \log m.$$

Moreover, $$n^m = (a^x)^y = a^{xy};$$

$$\therefore \log n^m = xy = m \log n.$$

Likewise $\log n^{-m} = -m \log n,$

" $\log n^{\frac{1}{m}} = \log \sqrt[m]{n} = \frac{1}{m} \log n.$

Apply these ideas. We can write at once that to the base 10

$$\begin{aligned}
\log 5000 &= \log 5 + \log 1000 = 3 + \log 5,\\
\log 500 &= \log 5 + \log 100 = 2 + \log 5,\\
\log 50 &= \log 5 + \log 10 = 1 + \log 5,\\
\log 5 &= \log 5 + \log 1 = 0 + \log 5,\\
\log \tfrac{5}{10} = \log \cdot 5 &= \log 5 - \log 10 = -1 + \log 5,\\
\log \cdot 05 &= \log 5 - \log 100 = -2 + \log 5,\\
\log \cdot 005 &= \log 5 - \log 1000 = -3 + \log 5.
\end{aligned}$$

If we know log 5 therefore we should know the logarithm of five times any power of ten, or even of five times any root of ten; for

$$\begin{aligned}
\log 5\sqrt{10} &= \log 5 + \tfrac{1}{2} \log 10 = \cdot 5 + \log 5,\\
\log 5\sqrt{1000} &= \log 5 + \tfrac{1}{2} \log 1000 = 1\cdot 5 + \log 5,\\
\log \tfrac{5}{\sqrt{10}} &= \log 5 - \tfrac{1}{2} \log 10 = -\cdot 5 + \log 5,\\
\log 5\sqrt[3]{10} &= \log 5 + \tfrac{1}{3} \log 10 = \cdot\dot{3} + \log 5;
\end{aligned}$$

but this is perhaps hardly worth stating.

How are we to find log 5? We can, if we choose, express it by means of log 2, thus :

$$\log 5 = \log \tfrac{10}{2} = \log 10 - \log 2 = 1 - \log 2,$$

or $\log 2 + \log 5 = 1.$

Similarly $\log 20 + \log 50 = 3,$

$\log 20 + \log 5 = 2.$

So also

$$\begin{aligned}
\log 2 + \log 6 &= \log 12,\\
\log 3 + \log 4 &= \log 12,\\
\log 7 + \log 9 &= \log 63,\\
\log 8 + \log 8 &= \log 64,\\
\log 9 + \log 9 &= \log 81,\\
\log 17 + \log 13 &= \log 221,
\end{aligned}$$

$$\begin{aligned}
\log 6 - \log 2 &= \log 3, \\
\log 9 - \log 3 &= \log 3, \\
\log 4 - \log 3 &= \log 1\cdot\dot{3}, \\
\log 5 - \log 2 &= \log 1\cdot5, \\
\log 5 - \log 3 &= \log 1\cdot\dot{6}, \\
\log 7 - \log 5 &= \log 1\cdot4, \\
\log 9 + \log 16 &= \log 144, \\
2 \log 12 &= \log 144, \\
3 \log 12 &= \log 1728,
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2} \log 12 = \log \sqrt{12} = \log 2\sqrt{3} &= \log 2 + \log \sqrt{3} \\
&= \log 2 + \tfrac{1}{2} \log 3,
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2} \log 16 &= \log 4, \\
\tfrac{1}{3} \log 8 &= \log 2, \\
\tfrac{1}{2} \log 49 &= \log 7, \\
\tfrac{1}{2} \log 25 &= \log 5,
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2} \log 72 &= \tfrac{1}{2} \log (36 \times 2) = \log 6 + \tfrac{1}{2} \log 2 \\
&= \tfrac{1}{2} \log (9 \times 8) = \log 3 + \log 2\sqrt{2} \\
&= \log 3 + \log 2 + \tfrac{1}{2} \log 2.
\end{aligned}$$

This might have been set as an exercise. Prove that

$$\tfrac{1}{2} \log 72 = \log 3 + 1\cdot5 \log 2.$$

One way to prove it would be to double both sides,

$$\begin{aligned}
\log 72 &= 2 \log 3 + 3 \log 2 \\
&= \log 3^2 + \log 2^3 \\
&= \log 9 + \log 8 \\
&= \log (9 \times 8) \\
&= \log 72. \qquad \text{Q.E.D.}
\end{aligned}$$

Exercises.—Verify, by means of the curve in this chapter, the following approximate statements,

$$\log 2 = \cdot3, \quad \log 4 = \cdot\dot{6}, \quad \log 8 = \cdot9,$$

$$\begin{aligned}
\sqrt{10} &= 3\cdot16\ldots, & \sqrt[3]{10} &= 2\cdot15\ldots, \\
\sqrt{11\cdot6} &= 3\cdot4\ldots, & \sqrt[3]{115} &= 4\cdot86\ldots, \\
\sqrt{7} &= 2\cdot6\ldots, & \sqrt{841} &= 29\cdot0.
\end{aligned}$$

CHAPTER XX.

On incommensurables and on discontinuity.

By this time it should have struck pupils with any budding aptitude for science, and for such alone is this particular chapter written, that it is strange and rather uncanny, unexpected and perhaps rather disappointing, that magnitudes should exist which cannot be expressed exactly by any finite configuration of numbers: not only that they should exist, but that they should be common. Draw two lines at right angles from a common point, each an inch long; then join their free ends, and measure the length of the joining line (which is often called the hypotenuse of the right-angled isosceles triangle that has been constructed): that is one of the quantities that cannot be expressed numerically in fractions of an inch, *i.e.* in terms of the sides. Its value can be approximated to and expressed, say in decimal fractions of an inch, to any degree of accuracy we please; but the more carefully it is measured the more figures after the decimal point will make their appearance: the decimal is one that never stops and never recurs. An infinite number of digits are necessary for theoretical precision, though practically six of them would represent more accuracy than is attainable by the most careful and grown-up measurement. It is therefore incommensurable, and can only be expressed exactly by another incommensurable quantity, viz. in this case the square root of 2. The length is $\sqrt{2}$ times an inch, or about 1·4142 ... inches. Draw a square upon it and it will be found to be two square inches in area. That is just the fact which (when proved) enables us to assert that each of its sides is of length $\sqrt{2}$; since that is the meaning of the phrase "square root."

It may be proved by the annexed figure:

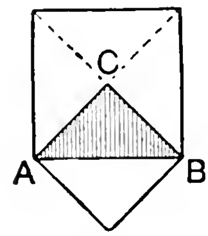


FIG. 11.

Where the shaded area *ABC* is an isosceles right-angled triangle, the area of which is repeated several times in the figure; four times inside a square drawn on the hypotenuse *AB*, and twice inside a square drawn on one of the sides *AC*. Wherefore the square on *AB* is twice the square on *AC*.

Observe however that there is nothing necessarily incommensurable about a hypotenuse itself: it is only incommensurable when the sides are given. It is easy to draw a hypotenuse of any specified length, say $1\frac{1}{2}$ inches long, and to complete an isosceles right-angled triangle; but now it is the sides that will be incommensurable. The real incommensurability is not a length, but a ratio, that is a **number** from which dimensions have cancelled out. No length is incommensurable, but it may be inexpressible in terms of an arbitrarily chosen unit, *i.e.* it may be incommensurable with the unit selected, and the chances are infinity to one that any length pitched upon at random will be in this predicament. It will not be precisely expressible in feet or metres, nor even in fractions of them, though it can be expressed with any degree of accuracy required.

The hypotenuse of most right-angled triangles will be incommensurable with both the sides, but there are a few remarkable exceptions; one in especial, known to the ancients, viz. the one where the sides are in the ratio of three to four. If such a triangle be drawn, with the sides respectively three inches and four inches long, the hypotenuse will be found to be five inches long; the more accurately it is measured the nearer it approaches to 5. It can indeed be shown theoretically, it is shown in Euclid I. 47, that it equals 5 exactly: a surprising and interesting fact.

With an isosceles right-angled triangle however, no such simple relation holds: the hypotenuse is $\sqrt{2}$ one of the sides, and $\sqrt{2}$ is incommensurable; for, as we have previously suspected and may now see, **every** root, whether square or cube or fourth or any other root, of every whole number, is incommensurable, unless the number be one of the few and special series of squares or cubes or higher powers. Cf. p. 175.

To prove this we have only to observe that:

The square or any higher power of a fraction can never be other than a fraction; for you cannot fractionate a fraction into a whole.

The square of a fraction cannot be an integer. Hence no integer can have a fraction as its square root.* Yet every integer must have a square root of some kind, that is a quantity which, squared or multiplied by itself, will equal the given number; but this quantity, though it may be readily exhibited geometrically and otherwise, can never be exhibited as a fraction, *i.e.* it cannot be expressed numerically by any means, either in vulgar fractions or in decimals or in duodecimals or in any system of numerical notation; in other words, every root of every integer except unity is incommensurable (incommensurable, that is, with unity or any

* Attend here. It is easy to miss the meaning.

other integer), except of those few integers which are built up by repeating some one and the same integer as a factor; for instance the following set:

$$4 = 2 \times 2$$
$$8 = 2 \times 2 \times 2$$
$$9 = 3 \times 3$$
$$16 = 4 \times 4$$
$$25 = 5 \times 5$$
$$27 = 3 \times 3 \times 3$$
$$32 = 2 \times 2 \times 2 \times 2 \times 2$$
$$36 = 6 \times 6$$
$$49 = 7 \times 7$$

and so on;

which class of numbers are therefore conspicuous among the others and are called square and cube numbers, etc. Every root of every other number is incommensurable, and most roots of these are too.

Not roots alone but many other kinds of natural number are incommensurable: circumference of circle to diameter, natural base of logarithms, etc., etc.; everything in fact not already based upon or compounded of number, like multiples, etc.

Incommensurable quantities are therefore by far the commonest, infinitely more common in fact, as we shall find, than the others: "the others" being the whole numbers and terminable fractions to which attention in arithmetic is specially directed, which stand out therefore like islands in the midst of an incommensurable sea; or, more accurately, like lines in the midst of a continuous spectrum.

What is the meaning of this? The meaning of it involves the difference between continuity and discontinuity. There is something essentially jerky and discontinuous about number. Numerical expression is more like a staircase than a slope: it necessarily proceeds by steps: it is discontinuous.

A row of palings is discontinuous: they can be counted, and might be labelled each with its appropriate number. Milestones are also discontinuous, but the road is continuous. The divisions on a clock face are discontinuous and are numbered, and, oddly enough, the motion of the hands is discontinuous too (though it need not theoretically have been so, and is not so in clocks arranged to drive telescopes). The hands of an ordinary clock proceed by jerks caused by the alternate release of a pair of pallets by a tooth wheel—an ingenious device called the escapement, because the teeth are only allowed to escape one at a time; and so the wheels revolve and the hands move discontinuously, a little bit for every beat of the pendulum, which is the real timekeeper. The properties of a pendulum as a timekeeper were discovered by Galileo; an escapement of a primitive kind, and a driving weight, were added to it by Huyghens, so that it became a clock.

Telegraph posts are discontinuous, but telegraph wires are continuous. They are discontinuous laterally so as to keep the electricity from escaping, but they are continuous longitudinally so that it may flow along to a destination.

But, now, are we so sure about even their longitudinal continuity? The pebbles of a beach are discontinuous, plainly enough; the sand looks a continuous stretch; but examine it more closely, it consists of grains; examine it under the microscope, and there are all sorts of interesting fragments to be found in it: it is not continuous at all. The sea looks continuous, and if you examine that under the microscope it will look continuous still Is it really continuous? or would it, too, appear granular if high enough magnifying power were available? The magnifying power necessary would, indeed, be impossibly high, but Natural Philosophers have shown good reason for believing that it, too, is really discontinuous, that it consists of detached atoms, though they are terribly small, and the interspaces between them perhaps equally small, or even smaller. But even so, are they really discontinuous? Is there *nothing* in the spaces between them, or is there some really continuous medium connecting them?

The questions are now becoming hard. Quite rightly so; a subject is not exhausted till the questions have become too hard for present answer.

There are several curious kinds of subterranean or masked continuity possible, which may be noted for future reference. Look at a map of the world; the land, or at least its islands, are after a fashion discontinuous, the ocean is continuous; but the land is continuous too, underneath, in a dimension not represented on the map, but recognisable if we attend to thickness and not only to length and breadth.

Human beings are discontinuous: each appears complete and isolated in our three-dimensional world. If we could perceive a fourth dimension, should we detect any kind of continuity among them?

The questions have now become too hard altogether; we have left science and involved ourselves in speculation. It is time to return. A momentary jump into the air is invigorating, but it is unsupporting, and we speedily fall back to earth.

But how, it may be asked, does this discontinuity apply to number? The natural numbers, 1, 2, 3, etc., are discontinuous enough, but there are fractions to fill up the interstices; how do we know that they are not really connected by these fractions, and so made continuous again? Well, that is just the point that deserves explanation.

Look at the divisions on a foot rule; they represent lengths expressed numerically in terms of an arbitrary length taken as a unit: they represent, that is to say, fractions of an inch; they are the terminals of lengths which are numerically expressed; and between them lie the unmarked terminals of lengths which cannot be so expressed. But surely the subdivision can be carried further; why stop at sixteenths or thirty seconds? Why proceed by constant halving at all? Why not divide originally into tenths and then into hundredths, and those into thousandths, and so on? Why not indeed? Let it be done. It may be thought that if we go on dividing like this we shall use up all the interspaces and have nothing left but numerically expressible magnitudes. Not so, that is just a mistake; the interspaces will always be infinitely greater than the divisions. For the interspaces have all the time had evident breadth, indeed they together make up the whole rule; the divisions do not make it up, do not make any of it, however numerous they are. For how wide are the divisions? Those we make, look, when examined under the microscope, like broad black grooves. But we do not wish to make them look thus. We should be better pleased with our handiwork if they looked like very fine lines of unmagnifiable breadth. They ought to be really **lines**—length without breadth; the breadth is an accident, a clumsiness, an unavoidable mechanical defect. They are intended to be mere divisions, subdividing the length but not consuming any of it. All the length lies between them; no matter how close they are they have consumed none of it; the interspaces are infinitely more extensive than the barriers which partition them off from one another; they are like a row of compartments with infinitely thin walls.

Now all the incommensurables lie in the interspaces; the compartments are full of them, and they are thus infinitely more numerous than the numerically expressible magnitudes. Take any point of the scale at random: that point will certainly lie in an interspace: it will not lie on a division, for the chances are infinity to 1 against it.

Let a stone—a meteor—drop from the sky on to the earth. What are the chances that it will hit a ship or a man? Very small indeed, for all the ships are but a small fraction of the area of the whole earth; still they are a finite portion of it. They have some size, and so the chances are not infinitesimal; one of them might get struck, though it is unlikely. But the divisions of the scale, considered as mathematically narrow, simply **could not** get hit accidentally by a mathematical point descending on to the scale. Of course if a needle point is used it may hit one, just as if a finger-tip is used it will hit several; but that is mere mechanical clumsiness again.

If the position is not yet quite clear and credible, consider a region of the scale quite close to one of the divisions already there, and ask how soon, if we go on subdividing, another division will come close up against the first, and so encroach upon and obliterate the space between them. The answer is never. Let the division be decimal, for instance, and consider any one division, say 5. As the dividing operation proceeds, what is the division nearest to it?

At first 4 of course,
then 4·9,
then 4·99,
then 4·999,
and so on.

But not till the subdivision has been carried to infinity, and an infinite number of 9's supplied after the decimal point, will the space between be obliterated and the division 5 be touched. Up to that infinite limit it will have remained isolated, standing like an island of number in the midst of a blank of incommensurableness. And the same will be true of every other division.

Whenever, then, a commensurable number is really associated with any natural phenomenon, there is necessarily a noteworthy circumstance involved in the fact, and it means something quite definite and ultimately ascertainable.

For instance:

The ratio between the velocity of light and the inverted square root of the electric and magnetic constants was found by Clerk Maxwell to be 1; and a new volume of physics was by that discovery opened.

Dalton found that chemical combination occurred between quantities of different substances specified by certain whole or fractional numbers; and the atomic theory of matter sprang into substantial though at first infantile existence.

The atomic weights are turning out to be all expressible numerically in terms of some one fundamental unit; and strong light is thrown upon the constitution of matter thereby.

Numerical relations have been sought and found among the lines in the spectrum of a substance; and a theory of atomic vibration is shadowed forth.

Electricity was found by Faraday to be numerically connected with quantity of matter; and the atom of electricity began its hesitating but now brilliant career.

On the surface of nature at first we see discontinuity, objects detached and countable. Then we realise the air and other media, and so emphasise continuity and flowing quantities. Then we detect atoms and numerical properties, and discontinuity once more makes its appearance. Then we invent the ether and are impressed with continuity again. But this is not likely to be the end; and what the ultimate end will be, or whether there is an ultimate end, are questions, once more, which are getting too hard.

CHAPTER XXI.

Concrete Arithmetic.

It is highly desirable that arithmetical practice should be gained in connexion with laboratory work, for then the sums acquire a reality, and interest is preserved. It is absolutely essential that all concrete subject-matter be based upon first-hand experience, for unless that can be appealed to, abstractions have no basis, but are floating unsupported in air. It far too frequently happens that a child, constrained to do sums expressed in terms of weights, has never weighed a thing in its life. It is the same mistake as is made when a child is drilled in the formal grammar of a language about which it knows absolutely nothing. In every case concrete experience should be the first thing provided, and abstractions may follow. The teacher is apt not to realise this, because grown persons have necessarily acquired *some* first-hand experience in the ordinary course of life; but a teacher who is really educated all round and has a living acquaintance with a great number of subjects should be able to enliven a lesson into something quite exciting, if only he or she can cultivate the patience necessary to allow time for the individuals of a class to attain some first-hand experience for themselves.

This is the real object of school laboratory work, and the mathematical teacher should seek to keep in touch with, and to be aware of, what the pupils are doing under other teachers, so as to illuminate his abstractions with concrete instances and examples. By far the best kind of examples are not those contained in books, but those which arise naturally or are invented by a stimulating teacher in the course of his exposition, or as a result of actual manipulation on the part of the taught.

The result of a laboratory measurement is *always* an incommensurable number; for the mere counting of a number of distinct objects is not to be called a laboratory measurement. No measurement of length, for instance, could ever be expressed as a whole number of inches, nor yet as a whole number plus a definite fraction of an inch. No measurement that ever was made could be expressed by either a terminating or a recurring decimal, nor by a vulgar fraction; for any of these modes of specification would imply infinite accuracy.

Suppose that an astronomical measurement is expressible by the number 17·4673, it is absolutely certain that 3 cannot be the last digit of the series if it is to be expressive of absolute fact. It may be that the next is 0, and perhaps the next also, but unless you can guarantee that all the digits to infinity are 0, the only reason for stopping at 3 (and it is a good reason) is that we can measure no more.

So a decimal expressing the result of measurement cannot terminate, neither can it recur. For, suppose the result, as nearly as we could get it, were 4·6666, how do we know that the next digit is going to be 6, and the next and the next also? We cannot know it.

If it did recur it would be the vulgar fraction $4\frac{2}{3}$; hence, this also is strictly an impossibly accurate result of measurement. The same with every vulgar fraction: it may be an approximate result, but no more.

The phrase a quarter, or a half, or seven-eighths, is appropriate therefore to rough specifications of approximate magnitude, but is inappropriate to precise specification of anything beyond *counting* of objects and fractions of an object. *Measurements* should be expressed in decimal notation, and the number of significant figures given should be characteristic of the order of accuracy of the work.

The meaning of significant figures and practical accuracy.

Rough workshop measurements are accurate, let us say, to 3 significant figures. Students' measurements in Physics, which are naturally more difficult than those of the workshop, if of the schoolboy kind, do well if they are accurate to two significant figures. For instance, if the latent heat of melting ice came out 79 or 80, it is quite as good as can be expected. A great deal of trouble is necessary to get a third figure right, for of course it means just ten times the accuracy. A good student would however try to get the third figure right, and might succeed, if it were not too complicated a measurement. The Demonstrator, and senior students who give some months to the work, would aim at 4 figure accuracy, and, if they attained it, would do well. A few exceptionally skilled experimenters with a genius for the work, devoting a year to a research, might attain 5 figure accuracy, but such accuracy as this is generally limited to the astronomical observatory, where the measurements are fairly simple and the theory of the errors to which instruments are necessarily liable has been studied for centuries. In taking the mean of a number of astronomical observations, even 6 figure accuracy is attainable, but beyond this it is extremely difficult to go.

The fundamental measurements that have to be made are the following:

length
time
angle
mass

and of these, oddly enough, *length* is by far the hardest to do accurately, though the easiest to do approximately.

Time is measured with considerable accuracy, even by a pocket watch. Suppose the watch were uncertain by 3 seconds a day, it would not be bad. If it lost or gained *regularly* it would be a perfect time keeper, for a regular loss can be estimated and allowed for; but that is not feasible except in elaborate chronometers carefully preserved. What is meant by the above is that having allowed for any known regular loss, it may lose or gain 3 seconds a day irregularly, so that to be quite safe we might consider it uncertain to the amount of plus or minus 3 seconds, or 6 seconds altogether. There are 86,400 seconds in a day, so the outstanding possible error would be 6 parts in 86,400, or 1 part in 14,400, or 7 parts in a hundred thousand, or ·007 per cent., and it would therefore be liable to cause a bad error in the fifth significant figure—an error which even slightly affects the fourth. Still for a cheap watch that is good performance, and means long hereditary skill on the part of makers of watches.

You could not hope to measure a mile with the same accuracy as you can measure the length of a day.

Angles are not very difficult to measure, because a number of disturbing causes have no effect on the divisions of a circle. If the weather gets warmer or colder, your yard and other measures change, and clock-pendulums and watch hair-springs change too; but though a circle expands and its divisions grow wider with heating, their *number* is not affected; the expanded circle is still divided into 360 equal parts or degrees. There is something essentially numerical about the divisions of a circle; and measurement of angle is subject to fewer disturbing causes than measurement of length.

But the really easy thing to determine accurately is mass or weight. For it never changes whatever you do to it. The weight of a piece of matter is constant, whether it be hot or cold, or whether it be evaporated to a gas, or dissolved in a liquid, or whether it be molten, or boiled, or vaporised, or chemically decomposed, or burnt up, or subjected to any other operation. So far as is known, its weight continues absolutely unchanged; although in combustion it *appears* to increase in weight, because it combines with other things. Moreover the balance is an easy and an accurate instrument. Even a beginner can weigh on a reasonably delicate balance to 4 significant figures, that is, he could weigh ten grammes to the nearest milligramme. He could hardly do better than that.

It is possible, however, with elaborate care to weigh to 6 significant figures, *i.e.* to weigh 10 grammes to the hundredth of a milligramme; but it needs a good balance and precaution against currents of air, dust, warmth of observer's body, accidental electrification, in some cases, and other disturbing causes. These things do not really disturb either the weights or the thing weighed, but they disturb the balance.

However, this is a digression, so as to make clear what is meant by a reasonable number of significant figures. We see that the number that is properly to be recorded will depend upon circumstances, that every additional figure expressed is a claim to greater accuracy, and that it is always better to *aim* at too many than too few; but we should cultivate an instinct for knowing when we have recorded as many as the experiment, or the observation, or the circumstances will justify.

CHAPTER XXII.

Practical manipulation of fractions when decimally expressed.

SINCE the results of all actual measurements yield incommensurable numbers, it is desirable to be able to deal with them freely. The present chapter will be considered very elementary, but it is inserted thus apparently out of place in order to emphasise the desirability of reintroducing familiar matter with variations, and also, more particularly, to uphold the doctrine that the other things treated are equally easy: ease is only a matter of use and custom.

In the easy manipulation of fractions there is much to be learnt, and considerable practice is necessary to attain facility. It is not worth while to exaggerate this practice, because the resulting art is not an accomplishment capable of giving pleasure to other people, like some other arts which can be attained by practice; nevertheless, some practice in arithmetic is essential, and on this part of the subject some of the time which has been saved from hogsheads and drachms can be usefully and interestingly expended.

First of all, we may notice that the manipulation of fractions is much simplified when they are stated in the ordinary arithmetical notation, utilising the same system as is employed for whole numbers. The ease conferred is similar to that gained by abolishing strange denominations of every kind.

Thus it is simpler to deal with 17·34 cwts. than it is to deal with it when expressed as 17 cwts. 1 quarter, 10 pounds, 1 ounce, $4\frac{12}{25}$ drachms, which is the way that helpless children are constrained to deal with it.

So also it is simpler to deal with a fraction when expressed as 4·4 inches, than when expressed as 25 "mils" more than 4 inches and 3 eighths, or $4 + \frac{3}{8} + \frac{25}{1000}$ inches, which is, however, the way the British workman seems to prefer to have it expressed—to the detriment of international engineering operations.

In other words, it is always simpler to express a thing numerically in a single denomination than to employ a multitude of denominations or denominators.

Even such a thing as $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ is more simply expressed as 1·9375, though still better as $2 - \frac{1}{16}$, or $\frac{31}{16}$. Simplicity is attained by use of a single denominator, whether sixteenths or tenths, or whatever it may be. It is the admixture of denominations or denominators that is troublesome.

So also the manipulation of fractions when expressed decimally is as easy as the manipulation of whole numbers. Care has to be taken about the position of the digits in either case, and the explicit writing of the decimal point almost makes the matter easier. The essential rule is, **keep the decimal points under one another, and they will then keep the places of the digits right.**

Thus, add together

$$4{\cdot}375 + {\cdot}025 + 53{\cdot}1.$$

The sum is written

```
  4·375
   ·025
 53·1
 ------
 57·500
```

and the result is verbally expressed as $57\frac{1}{2}$.

For just as various denominations, inches, weeks, months, ounces, tons, gallons, are handy in speech and for realising and speaking of magnitudes after they have been calculated, so vulgar fractions are often handy enough to express a result at the end. When they are complicated, however, they should only be used to quickly express approximate results. For instance, 5·12 inches might be spoken of as about $5\frac{1}{8}$th inches. So also 35·9 inches might be spoken of as about a yard. And the number 14·34 might be spoken of as about $14\frac{1}{3}$. For instance, if it expressed a length in feet, the length should be called 14 feet 4 inches if we were speaking to a carpenter. And similarly 5·67 feet would be approximately $5\frac{2}{3}$ feet, or 5 feet 8 inches.

In subtraction just the same rule holds: keep the decimal points vertical. *E.g.* to subtract 15·43 from 304,

write it

$$\begin{array}{r} 304{\cdot}00 \\ 15{\cdot}43 \\ \hline 288{\cdot}57 \end{array}$$

and there is nothing more to be said.

In countries with decimal coinage, this is all the arithmetic that book-keeping clerks have to employ. Although they may use the terms dollars, and quarters, and dimes, and cents in ordinary speech, they do not express a sum of money after our fashion, as

Dollars.	Quarters.	Cents.
17	3	18

but they express it simply as 17·93 dollars.

So, also, if another amount of 3 dollars, 2 quarters, and 17 cents has to be added, it is never expressed in that way, but as 3·67 dollars;

$$\begin{array}{r} 17{\cdot}93 \\ 3{\cdot}67 \\ \hline 21{\cdot}60 \end{array}$$

the addition is then quite easy, viz. 21·60 dollars. All addition becomes simple addition; and compound addition no more exists.

To express the resulting amount in the form given by compound addition (which try), viz. as 21 dollars, 2 quarters, and 10 cents would be unnatural; but it might, of course, be spoken of as 21 dollars and 60 cents, for that etymologically means precisely the same as 21 and 60 hundredths, *i.e.* 21·60.

The use of variegated and picturesque units, like weeks, and fortnights, and centuries, and acres, and hundredweights, and quarts, is to relieve the monotony of conversation; they should not be introduced into the workings of arithmetic. The end result can be interpreted into them, for vivid realisation, as occasion arises, and the instructed person should always be able to speak to the uninstructed person in his own language. For an instructed youth to expect workmen and others, who have not had his advantages, to appreciate his scholastic mode of expression, is barbarous, and shows a pitiful lack of sense on his part.

So long as popular units exist they should be employed in the proper place: they are part of folk-lore, and are often interesting enough; it is only when they are allowed to get out of their proper place and spoil the lives of children that they are to be condemned. In arithmetic proper they are out of place.

Now take multiplication. It is a little more troublesome, of course, but not much.

Keep the points vertical, as before; in other words, keep the digits expressive of the same denomination under each other, *i.e.* the units under the units, the tenths under the tenths, etc.; then the denomination of the answer looks after itself without any trouble.

For instance, multiply 30·57 by 4·3. Write it thus:

$$\begin{array}{r} 30{\cdot}57 \\ 4{\cdot}3 \\ \hline 122{\cdot}28 \\ 9{\cdot}171 \\ \hline 131{\cdot}451 \end{array}$$

I need not have written the last figure of the result, for most purposes; for since the data are only given to two places of decimals, an appearance of three decimal places in the result may give a notion of spurious and deceptive accuracy, and so is often better eschewed.

But this idea of approximate accuracy does not apply to results in pure mathematics, such as the properties of numbers, and things like that: it is the results of practical measurement that are not wanted to impossible accuracy, just as the price of a ship, or a railway, or a war, is not wanted closer than the nearest penny, if indeed so close.

I may say, however, that when we are dealing with the results of practical measurement, it is the number of significant figures in the whole specification, rather than the number of decimal places, which is the thing to be attended to. In the above sum the data involved four significant figures, and so a sixth significant figure in the result would be without meaning, and ought not to be written.

Now take a further example in multiplication: suppose we had 5·4306 grammes to multiply by 70·2: the whole sum would stand thus:

$$
\begin{array}{rl}
5{\cdot}4306 & \text{grammes} \\
70{\cdot}2 & \\
\hline
380{\cdot}142 & \\
1{\cdot}08612 & \\
\hline
381{\cdot}22812 & \text{grammes}
\end{array}
$$

The weighing was only given to 5 figure accuracy, so anything more is delusive in the result. Six figures may perhaps be permitted, that is as far as 381·228, but the last two figures after this, the 12, which are really ·00012, have no useful meaning, and need never have been written. And even the 8 is quite uncertain, so that the way to state the result with the same accuracy as the data is 381·23 grammes. *Three* being put as the last digit instead of *two*, because the next digit, viz. 8, carries it more than half way to the higher figure.

We observe then that when we multiply by a figure in the units place, we place the digits of the product under the corresponding digits of the multiplicand. When we multiply by a figure in the ten's place, we shift each digit one place to the left. If we multiplied by a figure in the hundred's place we should shift them two places to the left. Whereas when we multiply by a digit in the tenth's place, that is one place to the right of the decimal point, we shift the resulting figures of the product one place to the right, instead of writing them immediately under the corresponding digits of the multiplicand.

The rule about **division** is similar. Let us divide 470·82 by 5·7. Write it in its first stage:

```
5·7 ) 470·82 ( 8
      456·
      ————
```

Now, here $8 \times 5{\cdot}7 = 45{\cdot}6$, whereas in order to perform the subtraction we really require 456, else the decimal points would not be in the right position: hence the 8 is not really 8, but 80; that is it is not in the units' place, but in the ten's place, and so the decimal point is to be placed after the next digit.

Performing the subtraction indicated above, we see that the next digit of the quotient is a 2, and so the sum goes on without any further trouble or attention:

```
5·7 ) 470·82 ( 82·6
      456·
      ——————
       14·8
       11·4
       —————
        3·42
        3·42
        ————
```

And there happens to be no remainder. But, if there were, it would give no trouble; we should not take it up and express it as a vulgar fraction, but should continue the sum in the same way as before, bringing down ciphers as long as we chose, that is until we had got the quotient to the required degree of accuracy.

Dealing with fractions then in the decimal notation is just as easy as dealing with whole numbers in the same notation. The process is just the same, only we must be careful to put the decimal point in the right place. So, however, we must with whole numbers, only we do not have to actually write a decimal point in their case (except in the quotient perhaps); but we always have to be careful to *interpret* the quotient as meaning hundreds or thousands, or whatever it is, correctly, and that is essentialy the same thing as attending to the position of the decimal point.

For instance, divide 729 by 14.

```
14 ) 729· (52·07143
     70
     ---
      29·
      28·
     ----
       1·00
        ·98
       ----
        ·020
         014
        ----
           60
           56
           --
            40
```

The last figure in the quotient is not exactly 3, but that is the nearest, and it is quite time to stop, as we have already reached the extravagant accuracy of seven significant figures. If we wanted to go on, however, there is not the slightest difficulty. We simply go on till the remainder is negligible, not because it is itself numerically small, but because it occurs so many decimal places away from the left hand significant figure that only an utterly insignificant fraction is left. For instance, in the above sum, the last remainder which is indicated, as giving the quotient 3 in the fifth decimal place of the quotient, is really ·00040, and the multiplication of the divisor by 3 would give ·00042, which leaves a remainder of -2 in the fifth decimal place, to be divided by 14; with a result wholly trifling.

In the above sum the decimal points and a few preceding ciphers are indicated to show where they really occur, and to show how they might be indicated all the way along, if we chose; but there is no real need to indicate them anywhere except in the quotient. At the same time it sometimes helps to keep us right and clear to put all the points into the process, where they ought by rights to be, and always to see that they keep strictly vertical.

"Order" of Numbers.

As has been said before, in another connexion, p. 171, an extremely useful idea is the "order" of a number, that is to say the index of its order of magnitude: in other words, the power of 10 which it represents. This can be definitely specified by the distance of its highest significant figure to the left or the right of the unit's place: distances to the left being called positive, to the right negative; the unit's place itself being characterised by the order 0, and everything being reckoned from that as the zero position.

For instance, any single digit, like 6, would be of the order 0; 26 would be of the order 1; 526 of the order 2; 8526 of the order 3, and so on; the order being given by the *position* of the highest significant figure, and by nothing else. Thus 8526·79 would still be of the order 3; so would 8000, or 7000, or 1000.

26·79 is of the order 1.
6·79 is of the order 0.
·79 is of the order -1.
·09 is of the order -2.

What, then, is the order of ·00058? Here the highest significant figure is 5, and its position is 4 places to the right of the unit's place; hence the order of this number is -4. So also the numbers ·0001 and ·0009578 are of the minus-fourth order; but 1·0009 is of the order 0 again, and 27·0009 is of the order 1.

In the example ·00058 it is right to say that the digit 5 is of the order -4, the digit 8 of the order -5; and it is right to say that the number 58, which it contains, is also of the order -5. Again, in the number 525, we may say that the 52 which it contains is of the order 1, that is to say, that it occurs one place to the left of the unit's place.

It is often in practice convenient thus to attend to the order of particular digits, or pair of digits.

The rule for multiplication and division can now be given thus:

For multiplication of two numbers, take the highest significant figure of each, multiply them together, and give the resulting product a position representing the sum of the orders of the two digits taken. For instance, multiply 36 by 745. You take the two highest digits, 3 and 7, the sum of whose orders is $1 + 2 = 3$. The product, which is 21, has to be placed so that it shall have the order 3, that is to say, the unit's figure of the 21 is to be 3 places to the left of the unit's place.

Or take this example:—Multiply ·081 by ·742. We say 8 times 7 is 56, and this is to have the order compounded of -2 and -1, that is to say -3. Hence the 56 is to be placed so that its unit's digit is 3 places to the right of the unit's place ; or in other words, there will be one 0 between the 5 and the decimal point.

The rule for division is to be stated similarly :—

Take the first significant figure of the divisor, and the first one or two of the dividend : enough, that is, to be able to effect a division. Then the resulting quotient will have the order of this part of the dividend minus the order of the figure taken in the divisor.

For instance, if we had to divide 81 by 742, there would be no difficulty. We should take the 7 from the divisor, which is of the order 2, and 8 from the dividend of the order 1 ; and the quotient, has an order equal to the difference of the two orders, viz. – 1.

But if, on the other hand, we had to divide 742 by 81, we should take 8 from the divisor, where it is of order 1 ; but it would be useless to take 7 from the dividend : we must take 74, its place being also of the order 1 ; so that the resulting quotient will be of the order 0.

These matters are not particularly easy, they can be much simplified by employing powers of 10, as we will soon show ; but meanwhile we will do sums of this kind on commonsense principles, as follows : Divide ·742 by ·081. A simple and favourite way of doing such sums, is to get rid of the decimals as much as we please by shifting the decimal point in both equally, that is, multiplying them both by the same power of ten, so that it would be transformed into 742 ÷ 81 simply. The answer comes out about 9·1605.

One more example. Take the inverse of this sum.

```
·742 ) ·081 ( ·1
       ·0742
       -----
```

Here the first product 742 is required shifted one place to the right in order to come under the proper digits of the dividend, so the quotient must be not unity, but one tenth, or ·1. That once determined, the rest is quite ordinary.

```
·742 ) ·0810 ( ·109164
        ·0742
        -----
          6800
          6678
          ----
           1220
            742
           ----
            4780
            4452
            ----
             3280
```

Now, here it must be admitted that people clever at arithmetic do not write long division sums in so full and lengthy a manner. They do both the multiplication and the subtraction in their head, and write down the remainder only; so that the sum just done would look like this when people have done it by aid of the "shop" method of subtraction:

```
·742 ) ·0810 ( ·109164
          6800
           1220
            4780
             328
```

I can do it this way if I am put to it, but it seems to me a needless tax upon the brain, at least when grown up; and I am more likely to make mistakes and am less able to check them when made. Consequently for myself, I prefer the longer method, for it is the same sum in reality, the only difference is in the amount of it recorded on paper. I suppose that very clever people indeed would record nothing of it except the quotient: all the rest they would do in their head, as if it were a short division sum, or would even perceive, intuitively as it were, that $\frac{\cdot 0810}{\cdot 742} = \cdot 109164$. Boys have been known to be able to do things like this, and they are called calculating boys. They are, however, rather rare. Nevertheless, people when young are much cleverer at learning things than old folk, so perhaps they will get used to the abbreviated method of recording, if they begin young enough, and may like it better than the other. It is, I believe, found so.

One other point, however, I must not forget to mention here, and that is that if I had a sum like $\cdot742 \div \cdot081$ to do, I should first write it thus: $\frac{742}{81}$, and then proceed to look for factors. If they do not occur easily, it is not worth while to spend time in hunting for them; still less is it worth while to go through the farce of finding G.C.M. or H.C.F., or whatever it is called: one might as well be doing the long division sum as that. And then I should proceed to look out logarithms, and so turn it into simple subtraction. In the particular instance I have chosen, however, it is hardly worth while taking even this trouble, for directly you write 81, you see that you can divide by 9 in two stages; and although this might be found a little unsafe in old-fashioned times, when one had remainders to express as vulgar fractions, now that we know how never to be troubled with remainders, we proceed to divide numerator and denominator by 9 twice over, as follows:

$$\frac{742}{81} = \frac{82\cdot444\ldots}{9} = 9\cdot16049382716,$$

that is, for all practical purposes, $9\cdot1605$, as we found before (p. 207) by long division.

CHAPTER XXIII.

Dealings with very large or very small numbers.

But there is a mode of dealing with all these sums which is of great simplicity and service, and is more particularly useful when the figures to be dealt with are nowhere near the region of unity. In ordinary life we usually have to deal with a moderate number of things, or a few simple fractions of things; we seldom have to deal with billions or trillions, or with billionths or trillionths; but in science there is no restriction of this kind: we may have quantities of every order of magnitude to deal with. The human body is our natural standard of size, and on it our measuring units are or ought to be based. Everything much bigger than our body requires a large number to express it; so also anything incomparably smaller requires a very minute fraction to express it. We must be prepared to deal easily and familiarly with very large and very small numbers, and we need never suppose that a large number requires a great number of significant figures to express it; for by that means it would not be of any different size, it would only be expressed with preposterous accuracy. A number like 17,199,658 is for most purposes quite sufficiently expressed as 17·2 millions or 17,200,000.

So also our lifetime constitutes a natural human standard of time, and our walking and other movements are standards of velocity; but, to express the facts of nature in general, these magnitudes may have to be multiplied or subdivided to almost any extent. The distance of the fixed stars, and the velocity of light, and the age of the earth, are examples of one kind of magnitude. The size of atoms and the duration of their collisions lie towards the other end of the scale.

In many cases the precise numerical specification is of less importance than is the *order of magnitude*; sometimes because it is not accurately known, sometimes because it may be variable within certain limits. The "order of magnitude" may roughly be said to be given by the number of digits involved in its specification; in other words, by the power of ten concerned, without much regard to the particular figures that precede that power. Thus, for instance, in 3×10^{10} it is the index ten which gives the order of magnitude; the numbers 4×10^{10} and 5×10^{10} and even 8×10^{10} or 1×10^{10} are of the same "order," viz. 'ten.'

So also the numbers 30 and 70 are of the same order of magnitude, viz. 'one,' though one of the two numbers is more than double the other.

The closeness of specification required depends upon the subject matter and the object for which it is wanted. Occasionally, though not often, it would be possible to consider ten and a thousand as practically, though not technically, of the same order of magnitude: they would be roughly alike as compared with either a billion or a billionth.

Now let us take some examples of the *index* method of dealing with figures. Take first mere numbers of different orders of magnitude. For instance, divide, multiply, add, and subtract the following pair of numbers in every way:

$$a = 17{,}400{,}000, \quad b = \cdot 0015\,;$$

which may be called 17·4 millions, and 1·5 thousandths, or $17\cdot 4 \times 10^6$, and $1\cdot 5 \times 10^{-3}$ respectively.

First we notice that when numbers differ greatly in magnitude, addition and subtraction are operations that are useless; $a+b$ and $a-b$ are to all intents and purposes the same as a in the above case; the larger magnitude dominates the smaller, so far as addition or subtraction is concerned. A million plus or minus three is practically the same as a million. So no finite quantity added to infinity makes the smallest difference to it. This is a frequently useful fact: small quantities can be neglected when added to or subtracted from large ones.

$$1+z=1, \text{ when } z \text{ is small enough.}$$

$$a^2-x^2=a^2, \text{ when } x \text{ is small compared with } a,$$

which may happen either when a is very big or when x is very small, or even when both are big or both small so long as a is much bigger than x; in other words, so long as the ratio $\frac{x}{a}$ is small. The term "small," so used, signifies small compared with the other quantities concerned in the expression; or sometimes, as in this case of the ratio, small compared with unity.

But when we proceed to multiplication or to division, we find a very different state of things; there is then no domination of a big quantity over a small one; the bigness may be exaggerated, or it may be partially destroyed, by the influence of the small one.

Take the example suggested above:

$$ab=17{\cdot}4\times10^6\times1{\cdot}5\times10^{-3}=26{\cdot}1\times10^3=26100 \ldots\ldots\ldots\ldots(1)$$

$$\frac{a}{b}=17{\cdot}4\times10^6\div(1{\cdot}5\times10^{-3})=\tfrac{2}{3}\times17{\cdot}4\times10^9=11{\cdot}6\times10^9 \ldots\ldots(2)$$

$$\frac{b}{a}=\frac{1{\cdot}5\times10^{-3}}{17{\cdot}4\times10^6}=\frac{10^{-9}}{11{\cdot}6}=\frac{10^{-10}}{1{\cdot}16}={\cdot}862\times10^{-10} \ldots\ldots\ldots\ldots(3)$$

These results are numbered (1) (2) (3) for reference. The three results are of different orders of magnitude. The middle result is about half a million times the first; it is so much greater because a number has been divided by the small quantity b instead of being multiplied by it. The ratio between results (1) and (2) is therefore exactly equal to b^2, that is $(1{\cdot}5)^2 \times 10^{-6}$ or $2{\cdot}25 \times 10^{-6}$ or $\cdot00000225$.

The first result is more than a hundred billion times the third; the ratio between them being a^2, that is

$$(17{\cdot}4)^2 \times 10^{12} = 302{\cdot}76 \times 10^{12} = 3 \times 10^{14} \text{ approximately,}$$

an enormous number, but not bigger than what we have frequently to deal with in physics. The particles in a candle flame are quivering with about this number of vibrations per second, otherwise we should not be able to see the light. Everything self-luminous must be quivering at this or at a somewhat greater rate, consequently such rates of vibration are quite common.

The result (2) compared with result (3) shows a still greater difference in order of magnitude. To express the ratio, which is $\frac{11{\cdot}2}{\cdot862} \times 10^{19}$, a number of 21 digits is required, viz. the number $1{\cdot}3 \times 10^{20}$, more accurately $1{\cdot}2993 \times 10^{20}$, a number which is of the same order of magnitude as the number of atoms in a drop of water.

Now take another example. If light travels a distance equal to seven and a half times round the world in a second, how long does it take to come from the sun, a distance of 93 million miles? How long does it take to travel 1 foot, or say 30 centimetres. And how long to travel from molecule to molecule in glass, supposing that they are the ten millionth of a millimetre apart?

The circumference of the earth is just 40 million metres, by the definition of a metre. It therefore equals 4×10^9 centimetres. $7\frac{1}{2}$ times this equals 3×10^{10} centimetres; and this distance traversed per second gives the velocity of light.

A mile is about $1{\cdot}6$ kilometres, so the distance of the sun is $93 \times 1{\cdot}6 = 149$ million kilometres, as nearly as it is at present known; in other words it is $149 \times 10^6 \times 10^5$ centimetres $= 1 \cdot 49 \times 10^{13}$ centimetres. Hence the time required by light for its journey from the sun is

$$\frac{1 \cdot 49 \times 10^{13} \text{ centimetres}}{3 \times 10^{10} \dfrac{\text{centimetres}}{\text{second}}} = \frac{1490}{3} = 497 \text{ seconds},$$

or about 8 minutes and a quarter.

This begins to illustrate the right method of dealing with *units*. We shall have occasion to illustrate and emphasise it later at much greater length; but it will be seen already that the "centimetres" in numerator and denominator cancel out, and that the "seconds" in a denominator of the denominator come up to the top, and gives us the units of the answer. If this is not clear, never mind, we shall return to it and to much more like it. We might have written the whole working thus:

$$\frac{93 \text{ million miles}}{7 \cdot 5 \times 40 \text{ million metres per second}} = \frac{93 \qquad \text{miles}}{300 \text{ metres per second}}$$
$$= \cdot 31 \times 1600 \text{ seconds} = 496 \text{ seconds}$$
$$= 8\tfrac{1}{4} \text{ minutes approximately},$$

and that is really the best and safest way to do it. We have here put the actual data into the fraction, and then cancelled out the "millions"; next expressed numerically the ratio of miles to metres, which is 1600, since 1·6 kilometre is a mile; and then we bring the "per second" out of the denominator, and call it "seconds" in the numerator.

In so far as the two answers are not identical to the nearest second, that is simply because of the approximate working, which is justified by reason of the uncertainty of the data. If the result were expressed as 496·666 seconds it would be merely dishonest. The velocity of light and the distance of the sun are both quantities which have had to be experimentally determined, and neither is known with more than three figure accuracy. In fact, the latter is not known quite so closely as this. Moreover, it is at best only an average value: the sun is not always at the same distance from the earth, since the earth's orbit is not circular; the distance we have chosen is an approximation to the mean or average distance.

Now take the latter parts of the question, viz. the time required by light to travel a foot, or say 30 centimetres; and the time required to travel a molecular distance of the ten millionth of a millimetre, or 10^{-8} centimetre.

These are quite easily ascertained, since the velocity is given as 3×10^{10} centimetres per second. To travel 30 centimetres light takes $\frac{30}{3 \times 10^{10}}$ second, that is 10^{-9} second, which means the thousand millionth part of a second.

To travel from molecule to molecule, it takes $\frac{10^{-8}}{3 \times 10^{10}} = \frac{1}{3} 10^{-18}$, or say the third part of the trillionth of a second. Here the digit 3 is quite unimportant. The order of magnitude is all that is of any use, and that is the trillionth or 10^{-18} of a second. Molecular magnitudes are not known more accurately than that. It may be considered remarkable that they have been measured at all. The way they are obtained, a way necessarily indirect, can only be understood later. With attention to these early stages, this and much else can presently be understood by everybody. At present, grown people are ignorant of all these things, because they have not prepared their minds.

Now take a more childish example, akin to the horseshoe nails, page 155, and perhaps equally surprising.

A country the size of England was being besieged by a hostile fleet, and its inhabitants were in danger of starvation because they did not grow their own corn. Under these circumstances the captain of a merchant steamer craved permission from the enemy to run the blockade with a chess board full of wheat for his starving wife and family, the board to contain a single grain of wheat on the first square, two grains on the next, four on the next, and so on.

But when the enemy's admiral had had the necessary calculation made, by a Japanese sailor who happened to be on board, and was informed that the corn thus to be passed through his lines was sufficient not only to feed but to smother every living soul in the country, in fact to cover the whole land with a layer of grain more than a dozen yards thick, he declined to grant the request unless the whole supply were delivered at one operation.

To do the sum, proceed as follows:—The number of grains is 2^{64}, or, strictly speaking, one grain less than this number. A mode of arriving at this, if it is not obvious, will be given below, but it could be reasoned out by an intelligent beginner.

Call the number n.

then $$\log n = 64 \log 2,$$

and log 2, either from the curve (p. 179) or from a table of logs, is approximately ·3, more accurately ·30103;

hence $$\log n = 19{\cdot}266.$$

n is therefore a number with twenty digits of "order" 19; in other words, it is approximately eighteen trillion; more accurately it is $1{\cdot}845 \times 10^{19}$.

This number is not so great as the number of atoms in a drop of water, but it is a large number. To see what it means: buy half a pound of wheat as imported, without its husk, etc.—it costs only a penny—and devise a plan of practically counting the grains in say a cubic inch, without actually counting so many individually. This should not be beyond a youth's ingenuity.

I find that on the average a grain is $\frac{1}{4}$ inch long and $\frac{1}{7}$ of an inch broad. So if they were regularly arranged, in what is called square order, there would be 28 of them lying in a square inch; and if piled up an inch high, also in regular order, there would be $7 \times 28 = 196$ in the cubic inch so constructed.

An allowance for irregularity should doubtless be made, but it is uncertain; it is not even quite clear whether more or fewer could be got into a given space by a higgledy piggledy arrangement than regular packing in artificially square order. It will be near enough if we take it as about the same, and so estimate 200 grains of wheat to the cubic inch.

We are now prepared to go on with the sum set. The area of a country the size of England is given in the geography books, or the Penny Cyclopædia, as 50,000 square miles. A mile is 1760×36 inches $= 6{\cdot}336 \times 10^4$ inches, so a square mile is the square of this, viz. $40{\cdot}145 \times 10^8$ square inches, of which the first two digits are sufficient for our purpose.

Hence the area of a country as big as England is $5 \times 10^4 \times 40 \times 10^8 = 2 \times 10^{14}$ square inches. Now the number of grains which are to be distributed over this area is given by our previous working as $1{\cdot}845 \times 10^{19}$, and we have ascertained that, roughly speaking, 200 of the grains will occupy a cubic inch. Hence the number of cubic inches which have to be provided to hold all the corn, is the 200th part of $1{\cdot}845 \times 10^{19}$, that is to say, $\cdot 922 \times 10^{17}$; or just less than the tenth of a trillion cubic inches. To provide this capacity on the surface of the country, the grain would have to be spread all over it in a uniform layer, of thickness

$$\frac{\cdot 922 \times 10^{17} \text{ cubic inches}}{2 \times 10^{14} \text{ square inches}} = \cdot 461 \times 10^3 \text{ linear inches.}$$

In other words, the corn would flood the whole country to a depth of 461 inches, or 38·4 feet, which is as high as an ordinary house. All cottages would therefore be completely submerged by the chess board full of grain distributed uniformly over the face of the country.

Perhaps, however, our initial step, that the number of grains is precisely $2^{64} - 1$, was not obvious. It can easily be seen thus. The number on any square will be one grain more than all those on the preceding squares added together. Thus, for instance, the number on the third square is 4, and the two previous squares contain one grain and two grains respectively, or 3 grains together; which is one less than the number on the next. The number on the next following square is 8, and the three previous squares together hold 7, or again one grain less, and so on. Hence the number on the tenth square would be the number on the nine previous squares added together, plus one. The number on the tenth square is 2^9, so the number on the 9 previous squares added together will be $2^9 - 1$. The number on the sixth square is $2^5 = 32$, hence the number on the five previous squares added together will be 31; and so it is, viz. $1 \times 2 \times 4 \times 8 \times 16$, which equals 31. Compare page 323.

Now the total number of squares in a chess board is 8^2, or 64; the number of grains on a 65th square, if there were one, would be 2^{64}, hence the number on the 64 previous squares added together (which is just what we want) is $2^{64} - 1$.

This peculiar result of continued doubling, that the product each time just exceeds the sum of all the preceding products, has suggested a plan of what is called "breaking the bank," at a place where you stake on one of two events, either of which is equally probable, say red or black, and win back double your stake if you win, that is receive your stake and another added to it by the "bank." The simplest rule for "breaking the bank" is simply this: Begin small, and double your stake every time you lose; whenever you win, begin again.

If it were feasible to continue this process you could never really lose, because your stake would always just exceed the sum of your previous losses, so that whenever you won you would get them all back plus 1 counter more. Winning would be slow but sure. To work the process you must be prepared, however, with a considerable number of counters to stake with, if you happen to lose many times in succession. And as a matter of fact every 'bank' protects itself against so simple an arithmetical device by declining to receive more than a certain maximum stake. If therefore you have staked the maximum and lost, you have no way of getting your losses recouped; and so it is universally conceded, even by gamblers, that there are more profitable, as well as more useful, ways of earning a living.

The operation of constant doubling is a particular case of what is generally called geometrical progression, and it is remarkable how rapidly we can thus reach enormous magnitudes.

Of course, if instead of doubling we treble or quadruple each time, the large result will be reached still sooner, but, as a matter of fact, any constant factor greater than 1, repeated often enough, will grow to any magnitude; whereas any constant factor less than 1 repeated often enough will obliterate or reduce to insignificance any initial magnitude.

Take an example. Let the factor be 1·1, that is one and a tenth. Multiply it by itself 20 times, so that the result is $(1{\cdot}1)^{20}$, whose value can be found by logarithms easily enough, thus:

Call it x, then $\log x = 20 \log 1{\cdot}1 = 20 \times {\cdot}0414 = {\cdot}828$, so $x = 6{\cdot}6$; showing that the initial rate of increase is slow. Look, for instance, at the geometrical progression or compound-interest curve on page 357, which is the same as the exponential curve on page 101 taken backwards, and note that it begins slowly. But continue the process until we have reached $(1{\cdot}1)^{100}$, whose logarithm will be $100 \times {\cdot}0414$, or $4{\cdot}14$, and already the result is 13,800. To get really large numbers with such a factor as 1·1, we should therefore have to repeat the operation very often indeed.

So also to reduce to insignificance by means of such a factor as ·9, we should have to repeat the multiplication very often: for let $(\cdot 9)^{100} = x$

$$\text{then} \quad \log x = 100 \log \cdot 9 = 100 \log \tfrac{9}{10}$$
$$= 100 (\log 9 - \log 10) = 100 \log 9 - 100$$
$$= 95\cdot 42 - 100 = -4\cdot 58 = -5 + \cdot 42 = \bar{5}\cdot 42,$$

wherefore $x = \cdot 0000263$,
a number which is of the "order" -5.

Illustrations of excessively rapid multiplication by geometrical progression occur in Natural History, where certain organisms are known to increase at a prodigious rate, this rate of increase being the cause of plagues, like a plague of locusts, or blight, or like certain kinds of disease. For suppose a parent insect laid and hatched a thousand eggs (which is indeed a very moderate number), and suppose each of these also hatched a thousand, and suppose each generation only required a month to come to maturity, and lived for a year; the number of descendants in the course of twelve months would be a thousand raised to the twelfth power, that is to say a number of the "order" 36, or 1 followed by 36 ciphers, or a trillion-trillion.

Some diseases are caused by the fission or splitting up of cells into two or more, which rapidly grow and split up again. In such case the rapidity of increase can be still more prodigious, because the time which need elapse between the splitting and the re-splitting of cells may be short.

It does not follow that these geometrical-progression rates of increase apply without qualification to every kind of population, nor to one whose needs are in excess of available supply. For some actual facts, however, see Appendix IV.

CHAPTER XXIV.

Dealings with Vulgar Fractions.

Having now exhibited the easy mode of dealing with fractions, we must proceed to the more difficult method where the division operation has not been performed, but is only indicated: the same sort of indication as has been used in algebra. For instance, to divide a by b you cannot really do it in algebra, you can only indicate it, as $a \div b$ or $\frac{a}{b}$. So also in arithmetic if one has to divide 3 by 4, we can, if we choose, do it, and write ·75 simply, but for many homely purposes it is sufficient to indicate it only, and leave it in the form of $3 \div 4$ or $\frac{3}{4}$. Fractions left like this are not so easy to deal with, but they usually apply to such simple magnitudes that they are simple enough. For anything complicated, however, they are unsuitable, and they must be simplified; moreover, as we have seen in Chapter XXI. on concrete arithmetic, they seldom occur in practical measurements; nevertheless we must learn to perform the fundamental operations upon vulgar fractions without having necessarily to reduce them first to a simpler form. One way of dealing with mixed units, such as cwts., quarters, and lbs., or pounds, shillings, and pence, is to reduce them all to some one denomination; but it would be rather stupid if we did not know how to treat them in any other way. So also one way of treating collections of vulgar fractions is to reduce them all to some one denomination, or to decimals; but we ought to learn how to manage them without this preliminary operation.

So we will proceed to illustrate by example some simplifying processes, first reminding ourselves of the fundamental operations of addition, subtraction, multiplication, division, involution, and evolution, applied to vulgar fractions.

No further explanation is needed beyond what has gone before, Chaps. III., IX., etc. For **addition**, make a common denominator and cross multiply.

$$\frac{x}{a}+\frac{y}{b}=\frac{bx+ay}{ab};$$

or another example

$$\frac{x}{a}+y=\frac{x+ay}{a}.$$

For **subtraction**, the same.

$$\frac{x}{a}-\frac{y}{b}=\frac{bx-ay}{ab};$$

or if given $\frac{x}{a}-y$, it equals $\frac{x-ay}{a}$.

For **multiplication**, multiply numerators together and denominators together.

$$\frac{x}{a}\times\frac{y}{b}=\frac{xy}{ab},$$

or, in the common case when one denominator only appears,

$$\frac{x}{a}\times n=\frac{nx}{a}.$$

For **division**, invert the divisor, and multiply

$$\frac{x}{a}\div\frac{y}{b}=\frac{x}{a}\times\frac{b}{y}=\frac{bx}{ay},$$

or

$$\frac{x}{a}\div n=\frac{x}{na}.$$

For involution, operate similarly on both numerator and denominator.

$$\left(\frac{x}{a}\right)^n = \frac{x^n}{a^n}.$$

For evolution, just the same; only we may write n as $\frac{1}{m}$ if it is a fraction, if we like,

$$\left(\frac{x}{a}\right)^{\frac{1}{m}} = \frac{x^{\frac{1}{m}}}{a^{\frac{1}{m}}} = \frac{\sqrt[m]{x}}{\sqrt[m]{a}}.$$

Numerical Verifications.

The simplest fractions of all, to deal with, are those which are not really fractions, but integers in disguise, like $\frac{12}{3}$ or $\frac{24}{8}$, and these serve for testing any operation easily and quickly.

If these two are added, for instance, the result according to the above rule is $\frac{12}{3} + \frac{24}{8} = \frac{96 + 72}{24} = \frac{168}{24} = 7.$

Subtracted, the result is

$$\frac{96 - 72}{24} = \frac{24}{24} = 1.$$

Multiplied, the result is

$$\frac{288}{24} = 12.$$

Divided, they become

$$\frac{12}{3} \times \frac{8}{24} = \frac{4}{3} = 1 \cdot 333\ldots.$$

Squared, the results are

$$\frac{144}{9} = 16, \text{ and } \frac{576}{64} = 9, \text{ respectively.}$$

Square-rooted, we get

$$\frac{\sqrt{(12)}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}} = 2 \text{ for one.}$$

and $$\frac{\sqrt{24}}{\sqrt{8}} = \frac{2\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3} \text{ for the other.}$$

Of course, in practice we should get rid of such pretence or imitation fractions, by expressing them as whole numbers at once, before beginning operations on them. But we may often have fractions not unlike them; composed, that is to say, of a whole number and a proper fraction in addition. Those fractions, larger than unity, are sometimes called "improper fractions," and when expressed as an integer + a fraction, they are called "mixed numbers"; but these terms are hardly ever used out of the schoolroom.

A pretended fraction like $\frac{12}{3}$ cannot be concretely exhibited to small children unless there are several things to be cut up. With several apples we can do it; for if we cut them all up into thirds, and then pick out 12 of the thirds, we shall find that we can build up with them 4 apples.

So also $\frac{13}{3}$ would give us 4 apples and $\frac{1}{3}$ of another; and we could not exhibit it properly unless we had 5 apples to start with.

These fractions are called improper fractions because they are not fractions of one thing, but fractions of a lot of things.

To exhibit a proper fraction like $\frac{3}{8}$ is easy, for we have only to cut an apple into half quarters and then remove five of these, leaving $\frac{3}{8}$ths of the apple behind.

It is from this point of view that vulgar fractions are simpler than decimals. There is always some good reason why a popularly employed nomenclature has been hit upon. For these simple things it is excellent: it is only when we come to complicated things that we find it rather difficult. In practice, however, difficult sums never occur in this form, and there is no reason for wasting the time and brains of children in simplifying unwieldy artificial complications; for these things may give them much trouble when young, whereas later, if they ever learn mathematics, they will experience none at all, even if they come across the most complicated of them.

CHAPTER XXV.

Simplification of fractional expressions.

We will attend now to certain simple operations which constantly occur in practice; it is easy to get accustomed to them and to take an interest in them, as in any natural exercise of intelligence. First of all we will take "cancelling," that is, striking out of common factors, a process in which useful ingenuity can be trained. Examples are better than precept, so try the following: Simplify

$$\frac{36 \times 108 \times \cdot 91}{17 \cdot 28 \times 65 \times 8 \cdot 1}.$$

Here we see at a glance that a lot of factors can be struck out, because

$$36 = 3 \times 12, \qquad 108 = 9 \times 12, \qquad 91 = 7 \times 13,$$

though that last is not so likely to be known, unless an extended multiplication-table has been learnt—a very useful accomplishment; moreover

$$1728 = 12^3, \qquad 65 = 5 \times 13, \qquad \text{and } 81 = 9^2.$$

As to the position of the decimal point, that is a matter that gives no trouble at all. The decimal point must always occur somewhere; it is understood and not written at the end of integers, but it is there all the time; and its influence can be attended to after the cancelling has been done. Of course we might shift it equally to the right in both numerator and denominator, and so get rid of its explicit appearance, and we shall do this ultimately, but we will not do that at present since we want to use it as an example; we will cancel factors as they stand, and leave the decimal points unchanged in position till the end; and the result, written down in practice in a few seconds without all this talk, is

$$\frac{3\times 9\times \cdot 07}{\cdot 12\times 5\times 8\cdot 1}=\frac{3\times \cdot 07}{\cdot 6\times \cdot 9}=\frac{\cdot 07}{\cdot 18}=\frac{7}{18}.$$

This will not simplify further, because 7 is a prime number and does not go into 18.

It would be very seldom useful to write the result as

$$\frac{1}{2\frac{4}{7}} \text{ or } \frac{1}{2\cdot 5714\ldots},$$

but it would often be useful to write it as

$$\frac{3\cdot 5}{9}=\cdot 3\dot{8}88\ldots \text{ or approximately } \cdot 4.$$

Take another example: one less likely to occur however, one of a *double fraction.*

$$\frac{\frac{1}{7}\times\frac{32}{9}\times\frac{543}{17}}{\frac{8}{119}\times\frac{21}{100}}.$$

Here we may cancel out factors among the numerator fractions, and likewise among the denominator fractions; but we must not cancel a factor in the upper numerator against a factor in the lower denominator. 119 contains the factor 17, and also the factor 7, being equal to 7 × 17; and obviously 543 contains the factor 3, since its digits divide by 3. 100, as usual, is useless for factor purposes.

So we re-write the fraction (with needless elaboration)

$$\frac{\frac{1}{1}\times\frac{4}{9}\times\frac{181}{1}}{\frac{1}{1}\times\frac{7}{100}}, \quad \text{or} \quad \frac{\frac{4\times 181}{9}}{\frac{7}{100}},$$

the middle line being drawn rather longer and stronger than the other two, so as to show that the upper of the two fractions is to be divided by the lower.

We have now to multiply the extremes and divide by the means, a convenient rule to remember, giving us

$$\frac{400 \times 181}{63}.$$

But a rule of this kind should never be *given*, it should be ascertained and, if possible, invented by the pupil. To invent a handy rule involves a little bit of original thought, and the opportunity for exercising that vital power should never be lost.

Hence we should not *at first* make the above convenient short cut, but proceed thus:

This means $$\frac{4 \times 181}{9} \div \frac{7}{100},$$

and is an example of division of fractions, so invert the second number and multiply

$$\frac{724}{9} \times \frac{100}{7} = \frac{72400}{63}.$$

To express this in decimals we might proceed thus, for although 9 is not a factor of the numerator it will be approximately one, and can at any rate be divided out, leaving

$$\frac{8044 \cdot \dot{4}}{7} = 1149 \cdot \dot{2}0634\dot{9}206349 \ldots.$$

Digression on recurring decimals.—No importance attaches to the notation of the superposed dot or dots for circulating or recurring decimals. Children may write as many of the recurring places as it amuses them to write. In practice, the result would not usually be wanted beyond the first ·2, which is nearly equivalent to six figure accuracy, since the next figure is a 0.

The interpretation of recurring decimals as vulgar fractions, with a certain number of nines and noughts in the denominator, is of no practical moment. It should be reserved for the more intelligent and irrepressible children, and by them it might be found out, with great advantage. Children who delight in finding out such things are on the way to acquire some of the powers and tastes of the pure mathematician.

The simplest case may, however, be known, and perhaps this amount of hint would be necessary even to sharp boys. A recurring decimal is a geometrical progression, with fractional common ratio, and extending to infinity.

Thus the commonest of all

$$\cdot\dot{3} \text{ or } \cdot 3333\ldots\ldots\ldots$$

means
$$\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\ldots\ldots\ldots,$$

the common ratio being $\frac{1}{10}$th.

Hence its sum, by the rule for G.P. [see Chap. XXXV.] is

$$\frac{a}{r-1}=\frac{3}{10-1}=\frac{3}{9}=\frac{1}{3}.$$

The precise value of the answer on p. 227 expressed as a vulgar fraction, though never really wanted, is well known by everybody (needlessly well known for so trivial and useless a thing) to be $1149\frac{206349}{999999}$. *(End of digression.)*

The fact that
$$\frac{\frac{a}{x}}{\frac{b}{y}} \text{ equals } \frac{ay}{bx},$$

as is proved by writing it

$$\frac{a}{x}\div\frac{b}{y}=\frac{a}{x}\times\frac{y}{b},$$

is worth remembering: most easily remembered as worded in the rule, multiply the extremes for the new numerator and the means for the new denominator.

The y may be called a double denominator, and we observe that it comes up into the ultimate numerator.

The rule for cancelling may also be similarly illustrated:

$$\text{In}\qquad \frac{\dfrac{na}{x}}{\dfrac{nb}{y}},\ \text{or in}\ \frac{\dfrac{a}{mx}}{\dfrac{b}{my}},$$

the n's and the m's cancel out.

But in such a fraction as this

$$\frac{\dfrac{ra}{x}}{\dfrac{b}{ry}},$$

the r's, so far from cancelling, appear in the result twice over, that is, squared; for it equals

$$\frac{r^2ay}{bx}.$$

The rule for cancelling in the case of double fractions therefore is: cancel common factors from alternate members in the double fraction, then deal with the extremes and means to attain the simplified result.

It may be preferred, and it may be safer, to perform the latter operation first, and so keep all the cancelling for an expression in simple fractional form; but either is a correct procedure.

CHAPTER XXVI.

Cancelling among units.

It is not only *numbers* that can so be cancelled: we may and often do, have fractions composed of concrete or physical quantities—quantities with length, and breadth, and thickness, and weight, and velocity, and other things. It will be found that cancelling can conveniently go on among these also.

Suppose we had the following ratio to interpret:

$$\frac{330 \text{ yards} \times 16 \text{ square yards} \times 77 \text{ lbs.}}{4 \text{ inches} \times \frac{1}{2} \text{ mile} \times \cdot 14 \text{ ton} \times 5 \text{ minutes}},$$

an experienced eye would see at once that the result was a velocity, *i.e.* that it could be expressed as so many miles an hour, or feet per second. And the working is on the following lines, though again the actual operation is much speedier than is the explanation of it:—

First we have the ratio of yards to inches, which is 36, and this is most conveniently and safely recorded by erasing the word "yard" and replacing it by "36 inches."

Next in the numerator we have square yards, and in the denominator we have a linear mile, which is 1760 linear yards, and that value is therefore conveniently substituted for "mile."

Then we have the ratio of tons to pounds, which is 2240; and we get left with one of the "yard" factors of the square yards uncancelled in the numerator, and with "minutes" uncancelled in the denominator. The result, before any cancelling is done, will be the following:

$$\frac{330 \times 36 \text{ inches} \times 16 \text{ yards} \times \text{yards} \times 77 \text{ lbs.}}{4 \text{ inches} \times \frac{1}{2} \times 1760 \text{ yards} \times \cdot 14 \times 2240 \text{ lbs.} \times 5 \text{ minutes}}.$$

Now we can strike out a number of units common to both numerator and denominator, and can at the same time do some numerical cancelling, of which we will indicate the steps sufficiently, noting that 11 is a factor of 1760, because the sums of its alternate digits are equal.

$$\frac{30 \times 9 \times 32 \times 11 \quad \text{yards}}{160 \times \cdot 02 \times 2240 \times 5 \text{ minutes}}.$$

Now, we see that 16 will cancel out with 32, and of course the ciphers can go from the 30 and the 160.

So we get it thus

$$\frac{27 \times 2 \times 11 \quad \text{yards}}{\cdot 02 \times 2240 \times 5 \text{ minutes}}.$$

So many yards by so many minutes; in other words, a velocity of so many yards per minute. How many?

$$\frac{27 \times 11}{\cdot 1 \times 1120} = \frac{297}{112} \text{ yards per minute.}$$

Here, perhaps, it is simplest to resort to long division, since no more factors are obvious: so we might leave the answer as 2·6518 yards per minute, which is a sort of racing snail's pace; or we might reduce it to other units. This last is a thing which often has to be done, and so no opportunity for showing the right way to do it must be lost at this early stage.

This is the easiest and only safe way:

$$\frac{297 \text{ yards}}{112 \text{ minutes}} = \frac{297 \times 3 \text{ feet}}{112 \times 60 \text{ seconds}} = \frac{297}{2240} \text{ feet per second;}$$

or again

$$\frac{297 \text{ yards}}{112 \text{ minutes}} = \frac{297}{112} \frac{\frac{\text{miles}}{1760}}{\frac{\text{hours}}{60}} = \frac{297 \times 60 \text{ miles}}{112 \times 1760 \text{ hours}}$$

$$= \frac{27 \times 3}{56 \times 16} = \frac{81}{896} \text{ miles per hour.}$$

It may be said that simple reductions like that can easily be done without writing them down fully. So they can, but they can easily be done wrong. Change of units is a subject extraordinarily easy to make a slip in, especially by multiplying where one ought to divide. It is at best a mechanical process, and it should be done mechanically; that is by a straightforward method which involves no delicate thought, and affords no loopholes for mistakes to creep in.

To check the above result, we can recollect that 4 miles an hour is about 2 paces or 6 feet per second; so that the ratio of the above two specifications for the same thing should be roughly as 3 to 2. And so it is; for the first is very roughly $\frac{1}{7}$ foot per second, while the second is roughly $\frac{1}{11}$ mile per hour; and the ratio of 7 to 11 is not very different from that of 6·7 to 10, which is $\frac{2}{3}$rds.

This rough-and-ready checking, in terms of anything that comes handy, and with quite rough approximation to the figures, is very useful, and, in real practice, wise; else we exhibit the ridiculous result of academic correctness in minutiæ, and commercially hopeless error in the order of magnitude; so that, for instance, a quantity pretending to be accurate to four or five significant figures may be all the time a thousand times too great or too small.

This is the kind of thing that always moves the practical man to legitimate and sarcastic mirth, because he could get nearer than that by his own untutored instinct and common-sense. People who have been elaborately tutored, but have not taken care for themselves of their own birthright of commonsense, are denominated "prigs," and their existence tends to bring education into contempt.

We must take another example of this cancelling of units, and we will take an instance of the occurrence of a double denominator in them. Suppose the following given:

$$\frac{15 \text{ cwt.} \times (32 \text{ feet})^2}{1080 \text{ grammes} \times 400 \text{ centimetres per second} \times \frac{1}{2} \text{ yard}}.$$

Here the experienced eye will see that the result must be a *time*, for every kind of unit will cancel out except the "per second" in the denominator. This is what I call a double denominator, for the *per* alone would put it in a denominator; so the result is that it comes up into the ultimate numerator.

To work the sum, proceed thus (with full elaboration shown, because it is an illustration):

$$\frac{15 \times 112 \text{ lbs.} \times 32 \times 32 \times (\text{feet})^2}{1080 \text{ grammes} \times \dfrac{4 \text{ metres}}{\text{second}} \times 1{\cdot}5 \text{ feet}},$$

$$\frac{1120 \text{ lbs.} \times 8 \times 32 \text{ feet}}{1080 \text{ grammes} \times \dfrac{1 \text{ metre}}{\text{second}}},$$

$$\frac{18}{17} \times \frac{454 \text{ grammes} \times 256 \text{ feet}}{\text{grammes} \quad \times 3{\cdot}28 \text{ feet}} \text{ seconds}$$

$$= \frac{18 \times 454 \times 256}{17 \times 3{\cdot}28} \text{ seconds.}$$

To work this out, either a slide-rule or logarithm-table would be advantageous. Suppose we take this as an opportunity for utilising a table of four-figure logarithms, and see what we get.

log 18 = 1·2553	log 17 = 1·2304
log 454 = 2·6571	log 3·28 = ·5159
log 256 = 2·4082	
6·3206	1·7463

Subtracting, we get 4·5743,
which is the logarithm of 37530 to four significant figures; and 37,530 seconds is therefore the answer.

This is equal to 10 hours $25\frac{1}{2}$ minutes; and so the complicated expression, involving many kinds of units, with which we started, represents really nothing more elaborate than the length of a working day.

These examples are rather dull and artificial; but to take a real example, which would lead to this kind of concrete result, would assume some knowledge of mechanics or physics. Suffice it to say that plenty of quite similar examples will occur when we get to real subjects like those, and, meanwhile, all that we can show is that they involve no difficulty of dealing with and interpreting. No admixture of units involves anything the least difficult: it only wants disentangling; and, in order to disentangle it securely and easily, the best plan is not to be afraid of writing out the thing at length, with all the factors present—both the numerical and the concrete units, or standards—and so gradually boil it down by a mechanical process involving no troublesome thought.

Whenever thought is necessary, it is to be exercised vigorously, but it should not be wasted over simple mechanical operations. Take thought once for all, learn good methods, and so economise thought in future. This is, indeed, the principle of any mathematical machine. Machines can be constructed, and are used, for performing really intricate mathematical operations; for analysing out the harmonic constituents of a tidal or other irregularly periodic curve, for instance. To devise such a machine required thought, and indeed genius, of the highest order: to work it, requires nothing beyond what an intelligent office boy can learn.

CHAPTER XXVII.

Cancelling in Equations.

One more detail concerning cancelling may here be mentioned. It relates to cancelling on either side of an equation. One side of an equation may be considered as divided by the other, and the result equated to unity, so that the rules for cancelling are easily deduced.

For instance in $$\frac{nx}{a}=\frac{ny}{b},$$

the n's may be cancelled, for it is equivalent to

$$\frac{\frac{nx}{a}}{\frac{ny}{b}}=1,$$

or $$\frac{nxb}{nya}=1, \text{ or } \frac{bx}{ay}=1, \text{ or } bx=ay;$$

as might have been seen at once by cross multiplying.

Suppose, for instance, that it had been written thus:

$$\frac{nx}{a}-\frac{ny}{b}=0,$$

it would have been the same thing; and the left hand of this equation might be reduced to a common denominator, with the subtraction performed as far as possible, by writing

$$\frac{bnx-nay}{ab}=0.$$

How comes it that this is the same thing as $bx = ay$? Because it may be written

$$\frac{n}{ab}(bx - ay) = 0,$$

and, in order that this may be true, one of the two factors must vanish, that is, must itself equal 0. For you cannot get zero by multiplying two finite quantities together. Hence either $\frac{n}{ab}$ must equal 0, which is in some cases possible, but is clearly not here intended; or else $bx - ay = 0$. And the latter cannot happen unless ay and bx are equal.

So we get this simple rule, that when an expression is equated to 0 any factor can be struck out, without having to be accounted for, *provided always that that factor is not itself zero.*

This last is a most tremendously important proviso, and its neglect may land you in the utmost absurdity. If we strike out a factor zero, from an expression equated to zero, we may be striking out the very and only factor which made it zero; the factor left behind may have any value whatever: the equation declines to tell us for certain what that value is, and we must not proceed to work on the assumption that it does. Similarly, if a zero factor is cancelled on either side of an equation, we can make no deduction concerning the equality or otherwise of the residual factors.

Caution.

This inequality of zeros is a matter of great importance, and I must proceed to illustrate it even at this stage, though we shall find plenty of instances later on.

Suppose an expression like this were given, from which to find x.

$$\frac{16\sqrt{(n^2-4)(x^2+b^2)}}{3x} = 0,$$

we should be quite safe in striking out 16 and likewise $\frac{1}{3}$ (viz. the 3 in the denominator), for these numerical factors are certainly not zero, so we should get left with

$$\sqrt{(n^2-4)}\,\frac{x^2+b^2}{x}=0.$$

Now, if we strike out the factor $\sqrt{(n^2-4)}$ and the factor $\frac{1}{x}$, we shall be left with the impossible result $x^2+b^2=0$.

Why impossible? Because it means that

$$x^2 = -b^2,$$

and the square of a real number, whether that number be positive or negative, cannot possibly be negative; for two similar signs multiplied together give a positive sign always;

$$-3\times-3=+9 \text{ just as much as } 3\times 3 \text{ does.}$$

What the equation suggests is that, under the circumstances, n must equal 2. It is the $(n-2)$ component of the (n^2-4) factor, and not the factor containing x, which is responsible for the zero value of the whole; and the equation tells us, therefore, nothing at all about the value of x.

I do not say that that is all that can be deduced from the equation, but that is all that lies on the surface.

To clinch the danger of striking out a factor, without at the same time recollecting the possibility that it may be itself the essentially zero factor, the following absurdity may be given.

To "prove" algebraically that $2=1$.

Let $x=1$, so that $x-1=0$,
then $x^2=1$, and $x^2-1=0$.
So $x^2-1=x-1$, since both equal zero,
wherefore $(x+1)(x-1)=(x-1)$.

Cancel out the factor $(x-1)$ from both sides, and we get left

$$x+1=1\ ;$$

but we knew all the time that $x=1$, therefore the left hand side is 2, and so $2=1$.

Instead of going through the above farce, it would be briefer to say $2 \times 0 = 0$;
divide both sides by 0, hence

$$2 = 1;$$

or instead of 2 you may put any quantity you please.

It is a point that may possibly require emphasis, so we will put it still more evidently:

It is undeniable that 7×0 is 0,
and also that 6×0 is 0,
if then it be argued that $\therefore\ 7 \times 0 = 6 \times 0$,

and that the 0 factor may be cancelled out, it seems to follow that $7 = 6$.

It is unsafe then to press the axiom that things which are equal to the same thing are equal to one another, to cover the case when "the same thing" is zero.

It is a question whether we have a right to say that $7 \times 0 = 6 \times 0$ at all, although they are both zero. It rather depends on what we mean by 0. It is certainly untrue to say that $\frac{0}{0} = \frac{7}{6}$ always, because clearly any numbers might be substituted for the 7 and the 6. Do not, however, assume that $\frac{0}{0}$ is gibberish. A meaning can be found for everything if you are patient and persevering. At any rate, we have no right to cancel out the zero factor which alone is responsible for the pretended equality $6 \times 0 = 7 \times 0$. Of course the expression does not in practice occur in this crude form, but it occurs in some masked form, such as

$$18\,(x^2 - 4) = 39\,(x - 2),$$

whence, cancelling out the common factor $3\,(x - 2)$, we get

$$6\,(x + 2) = 13, \text{ or } x = \tfrac{1}{6};$$

which may, however, be quite false, and is not at all a *necessary* consequence of the equation from which it is supposed to be deduced; it is a *possible* consequence, or "solution," but $x = 2$ is another, and may be the only real one.

CHAPTER XXVIII.

Further Cautions.

BEFORE leaving the subject of "cancelling," it may be well to append a caution concerning a small point which does sometimes give trouble to a beginner. The fractions so far chosen for simplification had both numerator and denominator composed of *factors*; in other words, numerator and denominator was each really a single "term": they were not composed of a number of terms united by the sign + or −. Compound fractions of this latter kind are more troublesome. In arithmetic they do not often present themselves in this form for simplification, because when they occur, the addition or subtraction can be so easily performed that naturally it is done before any process of simplification is thought of. But, in algebra, addition and subtraction are operations that cannot be *done*, they are only indicated. Indeed that is one of the chief advantages of algebra, that the operations to be performed are preserved intact and evident, and are not masked by the poor achievement of performance.

Suppose then we had $\frac{3n + 8x}{24nx}$,

the whole thing is full of factors, but we may not cancel any. If only the + were replaced by × we could cancel everything, and leave nothing but unity; but as it is, the fraction is already in its simplest form, unless indeed we choose to split it up into two fractions.

Why may we not cancel anything? Because a factor, in order to be cancelled, must apply to the *whole* denominator and to the *whole* numerator. In the above, there is no factor which applies to the whole of the numerator. So there we are stopped. Let us however resolve it into two fractions

$$\frac{3n}{24nx}+\frac{8x}{24nx},$$

and from each of these cancelling is easy, yielding

$$\frac{1}{8x}+\frac{1}{3n}.$$

This form may be preferable to the first given form, or it may not. It depends on what we want to do with it.

Suppose however we take another example, very like the first, but upside down,

$$\frac{36my}{4m+9y}.$$

Still no factors can be cancelled, for there is no factor common to the two terms of the denominator; but now we cannot even separate it into two fractions. The attempt is often made by beginners; they try to write it

$$\frac{36my}{4m}+\frac{36my}{9y}=9y+4m,$$

a splendid simplification certainly, but bearing no resemblance whatever to the originally given fraction of which it is supposed by the mistaken beginner to be a counterpart.

The mistake is so often made that it is worth numerical illustration.

Example (i). $\frac{24+7}{144}$. *Example* (ii). $\frac{144}{24+7}$.

The first can be split into two fractions

$$\frac{24}{144}+\frac{7}{144}=\frac{1}{6}+\frac{7}{144}.$$

The second can not be split up at all. It could be written, if it were worth while,

$$\frac{6}{1+\frac{7}{24}} = \frac{6}{1 \cdot 291\dot{6}}.$$

Of course both, being arithmetical, can be written

$$\frac{31}{144} \text{ and } \frac{144}{31} \text{ respectively,}$$

and that is just why these forms do not occur in arithmetic as they do constantly in algebra.

Is no cancelling ever to be done when a numerator or denominator contains more than one term? Certainly there is, if each term has a common factor. For instance,

$$\frac{na+nb}{n^2ab} \text{ or say } \frac{21+51}{1071}.$$

If the + were replaced by × the n^2 would cancel out altogether; but as it is, only one n cancels out, and the result is

$$\frac{a+b}{nab} \text{ or } \frac{7+17}{357}.$$

I have found beginners who thought that if they used the factor in the denominator to cancel one of the terms in the numerator, they could not use it likewise to cancel the other term; who would wish therefore to divide the 1071 by 9 instead of by 3, and to write it 119 in the result, because a 3 has been cancelled out of *each* term in the numerator, and therefore it looks as if a 9 should be cancelled from the denominator. But there is every difference between striking out a factor from each of two *terms*, and striking it out from each of two *factors*. The mistake arises in fact from a momentary confusion between + and × .

When the expression is $\dfrac{mx+my}{m^2z}$,

the result is $\dfrac{x+y}{mz}$;

but if the expression had been

$$\frac{mx \times my}{m^2z},$$

the result would be

$$\frac{x \times y}{z};$$

but just as the former expression with the + sign would hardly occur as such in arithmetic, so the latter with the × sign would hardly occur as such in algebra; it would be written

$$\frac{m^2xy}{m^2z}$$

and no shadow of doubt could arise. The doubt seems to occur only when there are several terms.

Take the case of more than one term in both numerator and denominator, like

$$\frac{a+b}{x+y}.$$

Can we split this up into two fractions? Certainly, but *not* into

$$\frac{a}{x}+\frac{b}{y};$$

the two fractions into which it splits up are

$$\frac{a}{x+y}+\frac{b}{x+y},$$

the whole denominator occurring in both.

Cautions of a slightly more advanced character.

There is another mistake often made by beginners later on; and we may as well mention it here, along with the other cautions. When we have a simple factor applied to two terms, like

$$n(a+b),$$

we may take away the brackets and apply it to each term, getting the equivalent form

$$na+nb.$$

But although this is legitimate with a factor, it is not legitimate with *everything* that can occur outside a bracket,—not legitimate with a symbol of operation for instance; neither is it legitimate with a square root or a logarithm.

Thus: $$n(a+b)=na+nb,$$
but $$\sqrt{(a+b)} \gtrless \sqrt{a}+\sqrt{b},$$
and $$\log(a+b) \gtrless \log a+\log b.$$

The sign $\gtrless$ is to be read "is not equal to" or "does not equal."

The two root expressions are quite different, and each is already in its simplest form. To illustrate numerically:

$$\sqrt{(4+9)} \gtrless 2+3,$$

for $\sqrt{13}$, so far from being 5, is something between 3 and 4; because $3^2=9$ and $4^2=16$, so $\sqrt{13}$ lies between them, and as a matter of fact is 3·6055513....

(Never imagine from the accidental repetition of some figure, like the 5 in this number, that it is going to "recur." A root cannot possibly be a recurring decimal, for, if it could, it would be a fraction, and therefore commensurable; and a root is always incommensurable, except when it is an integer. See Chapter XX.).

So again of course $$(a+b)^2 \gtrless a^2+b^2,$$
and $$(x-y)^3 \gtrless x^3-y^3.$$

As to $\log a+\log b$, so far from equalling $\log(a+b)$, we know that it equals $\log(a\times b)$, that is $\log ab$.

So also $$a^{x+y} \gtrless a^x+a^y,$$
but, instead, $$a^{x+y}=a^x\times a^y.$$

$\sqrt{a}+\sqrt{b}$ is by no means equal to $\sqrt{(ab)}$, although $\log a+\log b$ does equal $\log ab$; nor does $\cos x+\cos y$ equal either $\cos xy$ or $\cos(x+y)$; they are all different. So we learn to be cautious with symbols of operation and not to treat them as factors nor to treat them all alike. We have to be very cautious about the removal of brackets in their case, and must always be sure that we understand the meaning and value of the symbol outside them. Some operations can be treated in this way, and some cannot; and we must learn to discriminate.

Before long we shall find that a highly important operator, denoted by $\frac{d}{dx}$, can be treated in this way; so that

$$\frac{d}{dx}(u+v)=\frac{d}{dx}u+\frac{d}{dx}v.$$

And another operation denoted by $\int dx$ can likewise be so treated, so that

$$\int(u+v)\,dx=\int u\,dx+\int v\,dx\,;$$

but these things have to be proved, they must never be assumed; and the time for discussing them is not yet.

We may notice however that the familiar symbol of operation $\times$ is one that *can* be treated in this way

$$7\times(4+6)=(7\times 4)+(7\times 6),$$

whereas the symbol $\div$ can*not* be so treated

$$7\div(4+6)\neq(7\div 4)+(7\div 6).$$

Nor can the symbol $+$ be so treated. Anything which can be so treated is said to be subject to "the distributive law," that is it may, and indeed must, be distributed among all the terms.

There is another law, spoken of as the "commutative law," which is sometimes applicable and sometimes not; that is to say it applies to some things and not to others. It applies when the order can be inverted; for instance,

$a\times b$ has the same value as $b\times a$.

3 times 4 gives the same number, though it does not suggest the same grouping, as 4 times 3.

Similarly $a+b$ is the same as $b+a$,

but $a-b$ is not the same as $b-a$;

it is numerically equal but is opposite in sign: an important distinction.

Nor is $a \div b$ the same as $b \div a$, not even numerically equal. It is not *opposite*, but "reciprocal."

Again there is a permutative law:

$$c \times ab \text{ is the same as } a \times cb \text{ or } b \times ac,$$

so also $$x + y + z = y + x + z \text{ etc.};$$

and under certain circumstances, though not invariably,

$$a \frac{d}{dx} u = \frac{d}{dx}(au);$$

but $a\sqrt{b}$ is not the same as $\sqrt{(ab)}$, nor the same as $b\sqrt{a}$; the three things are in fact equal to $\sqrt{(a^2b)}$, $\sqrt{(a^2b^2)}$, and $\sqrt{(ab^2)}$ respectively.

The expression $n \log x$ is not the same as $\log nx$, it equals $\log(x^n)$.

CHAPTER XXIX.

ILLUSTRATION OF THE PRACTICAL USE OF LOGARITHMS.

(i). How to look out a logarithm.

BELOW is given the simplest table of logarithms that can be used. You can buy four-figure logarithms conveniently printed on a card, and perhaps you may prefer to use them at once, because four-figure logarithms are accurate enough for many practical purposes, and are handy in actual work. But to explain the method of using a table and the principle of it, without niceties and details, the annexed table will serve quite well.

You will find this table repeated at the end of the book, folded in such a way that it is easy to refer to.

Table of 3-figure Logarithms.

	0	1	2	3	4	5	6	7	8	9
1·0	000	004	009	013	017	021	025	029	033	037
1·1	041	045	049	053	057	061	065	068	072	075
1·2	079	083	086	090	093	097	100	104	107	111
1·3	114	117	121	124	127	130	134	137	140	143
1·4	146	149	152	155	158	161	164	167	170	173
1·5	176	179	182	185	188	190	193	196	199	201
1·6	204	207	210	212	215	218	220	223	225	228
1·7	230	233	236	238	241	243	246	248	250	253
1·8	255	258	260	263	265	267	270	272	274	277
1·9	279	281	283	286	288	290	292	295	297	299
2	301	322	342	362	380	398	415	431	447	462
3	477	491	505	519	532	544	556	568	580	591
4	602	613	623	634	644	653	663	672	681	690
5	699	708	716	724	732	740	748	756	763	771
6	778	785	792	799	806	813	820	826	833	839
7	845	851	857	863	869	875	881	887	892	898
8	903	909	914	919	924	929	935	940	945	949
9	954	959	964	969	973	978	982	987	991	996

The triple digits which occur throughout this table are the decimal parts of the logarithms of the numbers on the left and above. The decimal point is not printed, but it is always to be understood, and on taking out the triple figures a decimal point must always be written in front of them.

Now let us use the table to find a few logarithms. The most obvious record in the table is that

$$\log 1 = 0, \quad \log 1{\cdot}1 = {\cdot}041, \quad \log 1{\cdot}2 = {\cdot}079,$$
$$\log 1{\cdot}3 = {\cdot}114, \text{ etc.} \quad \log 1{\cdot}9 = {\cdot}279,$$
$$\log 2 = {\cdot}301, \quad \log 3 = {\cdot}477, \quad \log 4 = {\cdot}602,$$

and so on.

Next we have, by the use of the top row of figures combined with the left-hand column,

$$\log 2{\cdot}1 = {\cdot}322, \quad \log 2{\cdot}2 = {\cdot}342, \quad \log 2{\cdot}3 = {\cdot}362, \text{ etc.}$$
$$\log 3{\cdot}1 = {\cdot}491, \quad \log 3{\cdot}2 = {\cdot}505, \text{ etc.}$$
$$\log 4{\cdot}1 = {\cdot}613, \text{ etc.}$$

and so on.

For all these figures there is nothing more to do than just extract the logarithms from the table as they stand.

But now suppose we wanted the logarithm of 20 or 30.

We know that $\log 30 = \log 10 + \log 3 = 1 + \log 3$, hence look out $\log 3$, and write $\log 30 = 1{\cdot}477$.

Similarly $\log 20 = 1{\cdot}301,$

$\log 70 = 1{\cdot}845,$ and so on.

So all we have to do in that case is to prefix a 1 to the decimal point.

If we wanted the logarithm of 11 or 12 or 13 it would be just the same, we must prefix a 1 to the decimal point, so that

$$\log 11 = 1{\cdot}041, \quad \log 12 = 1{\cdot}079, \quad \log 13 = 1{\cdot}114.$$

Similarly

$$\log 21 = 1{\cdot}322, \quad \log 22 = 1{\cdot}342, \quad \log 31 = 1{\cdot}491, \text{ etc.}$$

1 is called the "characteristic" of any number of "order" 1.

Further if we want the logs of 100 or 200 or 300, we must prefix a 2 to the decimal point because

$$\log 300 = \log 100 + \log 3 = 2 + \log 3 = 2{\cdot}477.$$

Similarly $\log 110 = 2{\cdot}041$, $\log 120 = 2{\cdot}079$,

$\log 210 = 2{\cdot}322$, $\log 220 = 2{\cdot}342$,

$\log 310 = 2{\cdot}491$, $\log 320 = 2{\cdot}505$.

So the logarithm of any number consisting of two significant figures can be readily obtained from the table, and the "characteristic" or integer part of the logarithm is given by the "order" of the number. "Characteristic" and "order" are in fact two names for the same thing, except that the first is appropriate to a logarithm, and the second is applied to the original number.

The logarithm of every number with only two significant figures is therefore directly contained in the little table printed above, no matter how big the number may be. For instance,

$$\log 98{,}000{,}000 = 7{\cdot}991.$$

But suppose the number had 3 significant figures. What is the logarithm of 215 for instance? Well, it will lie approximately half-way between log 210 and log 220. Not exactly half-way because the number grows in G.P. while the logarithm grows in A.P., but half-way is near enough for most practical purposes. So we can see that approximately $\log 215 = 2{\cdot}332$, because that is half-way between 2·322 and 2·342.

But suppose the number whose log was wanted did not lie half-way between others, but only one-tenth of the way; suppose for instance log 211 was wanted, we should have to take one tenth of the difference between 322 and 342, which difference, being 20, the tenth of it is 2; and this would have to be added on, as representing one-tenth of the interval. So log 211 would equal 2·324.

We can in fact make an extension of the table for any third significant figure in the number whose log is required, thus,

$$\log 210 = 2{\cdot}322, \quad \log 211 = 2{\cdot}324,$$
$$\log 212 = 2{\cdot}326, \quad \log 213 = 2{\cdot}328,$$
$$\log 214 = 2{\cdot}330, \text{ and so on, up to}$$
$$\log 220 = 2{\cdot}342.$$

Take a few more illustrations of this.

Wanted log 2·35.

From the table, $\log 2{\cdot}30 = {\cdot}362$ and $\log 2{\cdot}40 = {\cdot}380$,

so $\log 2{\cdot}35 = {\cdot}371$.

Wanted log 3·41.

$$\log 3{\cdot}40 = {\cdot}532, \qquad \log 3{\cdot}50 = 544,$$

so $\log 3{\cdot}41 = {\cdot}533$ approximately.

Similarly $\log 3{\cdot}42 = {\cdot}534$ approximately.

Wanted log 5·63.

$$\log 5{\cdot}6 = {\cdot}748, \qquad \log 5{\cdot}7 = {\cdot}756,$$
$$\therefore \log 5{\cdot}63 = {\cdot}750,$$

being three tenths of the interval added on to the smaller one.

Wanted log 5·67. We might add on seven tenths of the interval to the smaller one, or, rather better, subtract three tenths of the interval from the bigger one, getting

$$\log 5{\cdot}67 = {\cdot}754.$$

But the table contains more than I have at present described and used. The first half of the table gives the logarithms of numbers near to unity, so we can get out logarithms of 1·01 or 1·02 etc. up to 1·99, the numbers being expressed to 3 significant figures and all the logarithms recorded. It is a help to have this given, as a sort of extra, because for these small numbers the logarithms change so rapidly that the jump is too great for easy and safe treatment by attending to the differences, and when we come to look out anti-logs (see next page) they will fall in gaps of too large size.

Using this part of the table we see that

$$\log 1{\cdot}01 = {\cdot}004, \qquad \log 1{\cdot}02 = {\cdot}009,$$
$$\log 1{\cdot}11 = {\cdot}045, \qquad \log 1{\cdot}25 = {\cdot}097,$$
$$\log 1{\cdot}53 = {\cdot}185, \qquad \log 1{\cdot}99 = {\cdot}299.$$

And consequently

$$\log 10{\cdot}2 = 1{\cdot}009, \qquad \log 102{\cdot} = 2{\cdot}009,$$
$$\log 111{\cdot} = 2{\cdot}045, \qquad \log 125{\cdot} = 2{\cdot}097,$$
$$\log 12{\cdot}5 = 1{\cdot}097, \qquad \log 19{\cdot}9 = 1{\cdot}299,$$
$$\log 199{\cdot} = 2{\cdot}299, \qquad \log 1990{\cdot} = 3{\cdot}299.$$

The characteristic of the logarithm is always the "order" of the number.

(ii). How to look out the number which has a given logarithm.

To look out the number which possesses a given log we have only to use the table backwards. It is quite simple and obvious in idea, the only trouble is that we shall not usually find the given logarithm actually in the table. If it is an extensive table we are more likely to find it, and that saves thought, but involves the turning over of many pages; with a little compressed table like the one given, we are not likely to find a number exactly entered, and a trifle of thought is necessary. That is no defect however for our present purpose, which is not immediately to facilitate practice, but to furnish instruction which shall facilitate practice by and bye.

The phrase "number which possesses the logarithm" so and so, is rather long and unwieldy, and it is commonly shortened into anti-log.

Thus $\log 2 = {\cdot}301$, or 2 is the anti-log of ·301.

Given then the following logarithm, ·380, what is its anti-log? Referring to the table, we see that it is 2·4.

Given ·663, the anti-log is 4·6, and so on.

But suppose the given logarithm had been 1·380, what then?

We should look in the table for the decimal part only, for it is only the decimal part which is ever there recorded. The prefix 1 before the decimal point tells us the power of ten in the result, shows in fact that the result lies between ten and a hundred. The antilog is therefore not now 2·4 but 24·

So also the antilog of 1·663 is 46· not 4·6.

The antilog of 2·663 is 460·,
of 3·663 is 4600·,

and so on. It is safer to actually write the decimal points at the end of whole numbers in this sort of case.

The integer part of the logarithm, often called its "characteristic," has simply the effect of determining the order of magnitude of the result (p. 171). Surely however a most important effect, and one not to be slurred over.

Examples.

What is the antilog of 1·672? Answer 47·.
What is the antilog of 1·301? Answer 20·.
What is the antilog of 2·041? Answer 110·.
What is the antilog of 3·699? Answer 5000·.

Employing the upper part of the table we see that

$\log 1{\cdot}34 = {\cdot}127$, $\log 1{\cdot}18 = {\cdot}072$,
$\log 1{\cdot}01 = {\cdot}004$ $\log 10{\cdot}1 = 1{\cdot}004$,
$\log 101 = 2{\cdot}004$, $\log 1010 = 3{\cdot}004$,
$\log 1{\cdot}09 = {\cdot}037$, $\log 10{\cdot}9 = 1{\cdot}037$,
$\log 109 = 2{\cdot}037$.

Likewise the antilog of ·111 is 1·29
" " 1·111 is 12·9
" " 2·111 is 129·
" " ·097 is 1·25.
" " 1·097 is 12·5.
" " ·196 is 1·57.
" " 2·196 is 157·.

So far the logarithms have been found in the table, because we chose numbers of only two significant figures. The case when a logarithm does not occur exactly in the table causes no difficulty, it only gives a little more trouble.

What is the antilog of ·389? Answer 2·45; because it lies about half-way between ·380 and ·398, so the answer lies half-way between 2·4 and 2·5.

What is the antilog of 2·389? Answer 245.

What is the antilog of 1·675? Answer 47·3.

Why? because the given log lies one-third of the way between 672 and 681; so its antilog will lie about one-third of the way between 47 and 48. As to the position of its decimal point, that is determined by the "characteristic" or integer part of the given logarithm, which was unity.

Logarithms of fractions.

So far all the antilogs have turned out greater than 1, because all the logarithms chosen have been positive. The characteristics have all been either 1 or 2 or 3 or 0; for the logarithm ·672 has the characteristic 0. It might be, and often is, written 0·672.

But now suppose it had a negative characteristic; for instance $\bar{1}$·672, where the minus sign is placed above the 1 instead of in front of it, in order that it may not be applied to the whole number, but only to the 1; which is a conventional but convenient mode of representing an important distinction.

The meaning, written out fully, is $-1 + {\cdot}672$.

Naturally this might be written, if we liked, $-{\cdot}328$, but if we did that we should want another table full of negative numbers wherein to look out the logarithms of fractions. By the above device we can use the same table all the time, and only adjust the position of the decimal point in accordance with the characteristic, so that it fixes the "order" as usual.

For instance,

antilog of ·672 from the table is 4·7,
so antilog of $\bar{1}$·672 will be ·47,
and antilog of $\bar{2}$·672 will be ·047,
antilog of $\bar{3}$·672 will be ·0047,

the negative characteristic indicating the position of the highest significant figure counting from the units' place.

The antilog of 1·672 is of course 47·
of 2·672 470·
and of 3·672 4700·

and so on; the positive characteristic counting the number of places to the left of the units' place.

(iii). How to do Multiplication and Division with Logs.

We know that $\log ab = \log a + \log b$,

and that $$\log \frac{b}{c} = \log b - \log c.$$

So we know that $$\log \frac{ab}{c} = \log a + \log b - \log c,$$

and likewise $$\log \frac{ab}{cde} = \log a + \log b - \log c - \log d - \log e$$
$$= (\log a + \log b) - (\log c + \log d + \log e).$$

In other words, whenever we have a fraction consisting of a number of factors in numerator and denominator, we must look out the logarithm of each factor. All those in the numerator, arrange in one column, and add; all those in the denominator, arrange in another column, and add; then subtract one addition from the other so as to get the logarithm of the quotient. We have then only to refer to the table to find the number which possesses this logarithm, and the quotient is found.

For instance, take this fraction

$$\frac{6{\cdot}7 \times 43 \times 170}{74 \times 3{\cdot}2 \times 1{\cdot}3}.$$

Look out the logs of the factors as stated:

log 6·7 =	·826	log 74· =	1·869
log 43· =	1·634	log 3·2 =	·505
log 170· =	2·230	log 1·3 =	·114
add	4·690	add	2·488
subtract	2·488		
	2·202		

2·202 is therefore the log of the resulting quotient. Referring to the table, we find that the number possessing this log is 159·2. Hence that is the answer.

It may be asked, why do it this way when we could easily do it by simple multiplication and division?

Reply: Very little, if any, advantage in such a simple case as that. No advantage at all if you can easily see factors which may be struck out.

But people who often have to do such sums get rather tired of frequent multiplication and division, and they usually prefer logarithms as a quicker and surer way. It becomes quicker and surer with practice. Engineers usually employ what is really the same process, but they have their table of logarithms constructed in wood; and instead of looking out the logarithms, they slide a slip along this rule, till a mark on it points to the number printed where its logarithm ought to be, and so attain the result in an ingenious manner, without actually recording or thinking of any logarithm at all. They shift the pointers, of which there are a pair, alternately to one factor after another, taking numerator and denominator factors alternately, and then at the end they read off the result as indicated by one of the pointers.

The instrument is called a "slide-rule"; it is in fact a mechanical table of logarithms arranged ingeniously for quick and practical use, and it gives you about 3-figure accuracy if it is of a simple and well made pocket kind. More elaborate and larger instruments can give 5-figure accuracy. The ingenuity belongs to the devising and making of the rule: the use of it is quite simple, but it has to be learnt. It should not be learnt as a *substitute* for other methods, but as a supplement. Pupils are not recommended to learn the slide-rule till they can use a numerical table of logarithms. Nor are they recommended to use logarithms till they can multiply and divide with facility. In other words, these aids to rapidity should be kept in their proper place,—not to make people helpless without them, but to assist people who can work quite well without them to obtain results more quickly and with less labour.

Another example. Find the value of

$$\frac{27{\cdot}1 \times {\cdot}16 \times {\cdot}089}{{\cdot}00055 \times 3430}.$$

Look out logs and record them as below:

$$\begin{aligned} \log 27{\cdot}1 &= 1{\cdot}433 \\ \log {\cdot}16 &= \bar{1}{\cdot}204 \\ \log {\cdot}089 &= \bar{2}{\cdot}949 \\ &\overline{\bar{1}{\cdot}586} \\ \text{subtract} \quad & 0{\cdot}275 \\ &\overline{\bar{1}{\cdot}311} \end{aligned} \qquad \begin{aligned} \log {\cdot}00055 &= \bar{4}{\cdot}740 \\ \log 3430{\cdot} &= 3{\cdot}535 \\ &\overline{0{\cdot}275} \end{aligned}$$

Antilog of this is ·205, which is therefore the result, and may be recorded as equal to the above fraction to something like 3-figure accuracy. This should be checked by actual multiplication. Indeed for some time, and especially when there are negative characteristics, it is safest to check over the result by other means than the mere logarithms. It is the order of magnitude that runs great risk of going wrong: the actual digits can easily be got right.

A few more examples for practice:

What is the log of ·05?

And what is the antilog of $\bar{1}·89$?

The log of 5 in the table is ·699,

so the log of ·05 is $\bar{2}·699$.

The antilog of ·89 estimated from the table is about 7·77,

so the antilog of $\bar{1}·89$ is ·777.

What is the antilog of ·049?

We find this number in the upper part of the table, and the antilog required is 1·12.

What is the antilog of 2·049? Answer 112.

" " $\bar{1}·049$? Answer ·112.

" " $\bar{2}·049$? Answer ·0112.

What is the antilog of ·023? It does not occur even in the upper part of the table, but it lies half-way between two numbers which we find there; so we estimate the antilog as 1·055.

The antilog of 3·023 = 1055·

of $\bar{1}·023$ = ·1055.

This is the use of the upper part of the table, as previously half explained, that it gives us the logarithms of numbers only slightly greater than 1 in greater profusion; and it is just here that profusion is necessary, for in other parts of the table logarithms lie much closer together in value than they do here. Consequently what would naturally be the first row of the table is spread out into nine rows, the first row itself becoming thereby a column, reaching from 1 to 2, and giving all the tenths of this interval.

If a beginner likes to think out the reason and meaning of the different closeness of distribution in various parts of a logarithm table, he should by all means do so, but he need not be made to do it. The reasonableness of it can be put thus:

All the 900 integers between 100 and 1000 have logarithms lying between 2 and 3: this unit difference of logarithms is therefore spread over all that range; while the same logarithmic interval, viz. that between 0 and 1, has to be squeezed between the numbers 1 and 10, covering only nine consecutive integers.

The logarithmic interval between 1 and 2 has to serve for the 90 whole numbers between 10 and 100, while the same logarithmic interval, viz. between 3 and 4, is all that can be allowed to cover the 9000 numbers between 1000 and 10000.

Hence manifestly the logarithms of integers between 1 and 10 must be few, and the intervals between must be great, though they may be conveniently filled up with the logarithms of intervening fractions; but the logarithms of integers between 1000 and 10000 are close together, their value increasing only slightly for each addition of unity to the number. In other words, the logs of integers take 9000 steps to go from 3 to 4; they only take 9 steps to go from 0 to 1. The one is a trip, the other is a straddle.

CHAPTER XXX.

How to find powers and roots by logarithms.

THE finding of any power, or any root, is now an extremely simple operation.

We know that $\log x^n = n \log x$, and this holds whether n be an integer or any fraction.

In other words, as said before,

$$\log x^2 = 2 \log x,$$
$$\log x^3 = 3 \log x,$$
$$\log \sqrt{x} = \log x^{\frac{1}{2}} = \tfrac{1}{2} \log x,$$
$$\log \sqrt[3]{x} = \log x^{\frac{1}{3}} = \tfrac{1}{3} \log x,$$

and so on. Hence the method suggests itself, and we need only proceed to examples.

To find the value of $\sqrt{2}$, the logarithm of it will be half the logarithm of 2, and that we look for in the table, and find to be ·301, so half of it is ·1505. This we do not find exactly in the table, but we see that it is the logarithm of a number lying between 1·41 and 1·42, and we estimate the number as being 1·414. This is of course only an approximation, because no arithmetical specification of it can be anything but approximate. If calculated more elaborately, it comes out 1·4142136..., but it can neither stop nor circulate.

Similarly $\sqrt{3}$ = 1·732 approximately,
or more nearly 1·7320508...,
again without either recurrence or termination.

Now take a case of a power. Suppose we want to calculate the 2^{24} involved in an example on page 156. Its logarithm will be $24 \log 2 = 24 \times \cdot 301 = 7\cdot 224$ and we have only to look out the number which has this logarithm, that is look for the antilog of 7·224. We shall find 225 in the table (p. 246), and that is really better than 224, because when we multiply a number by so big a factor as 24 there must probably be some carried forward figure to be attended to. Anyway we find that ·225 is the logarithm of 1·68. This is not the result, of course, since we have not yet attended to the characteristic, which is 7. The characteristic is indeed, in these big numbers, usually the most important thing to notice. The characteristic 7 shows that the number is of the order seven, *i.e.* that it lies between 10^7 and 10^8; in other words, that it requires eight digits to express it, and so it is approximately

16,800,000,

that is sixteen million eight hundred thousand, so far as we can express it with 3-figure accuracy. There are eight digits in this result, but only three of them are "significant," the others are mere ciphers to indicate the order of magnitude.

The neatest way of recording such a result is therefore

$$1\cdot 68 \times 10^7,$$

and the characteristic of the logarithm will always give us the index of the power of ten when the number is so written.

For instance,

$$\text{antilog}\, 19\cdot 330 = 2\cdot 14 \times 10^{19},$$
$$\text{antilog}\, \overline{6}\cdot 552 = 3\cdot 56 \times 10^{-6},$$
$$\text{antilog}\, \overline{2}\cdot 950 = 8\cdot 92 \times 10^{-2} = \cdot 0892.$$

The operation of finding a root will look thus:

To find the fifth root of 1930.

$$\log 1930 = 3\cdot 286,$$
$$\tfrac{1}{5} \log 1930 = 0\cdot 657 = \log \text{ of } 4\cdot 54\ldots$$

wherefore $4\cdot 54\ldots = \sqrt[5]{(1930)}.$

Observe that when dividing a logarithm the characteristic is to be included in it and divided with the rest of it. It is only in dealings with the "table" that the characteristic does not appear. It should however always be supplied and should not be forgotten or ignored.

Thus if we had wanted the fifth root of 193 or of 19·3 or of 1·93 we should have obtained a totally different number: not the same number with the decimal point shifted, but a different number altogether. For instance,

$$\log 193 = 2{\cdot}286,$$
$$\tfrac{1}{5}\log 193 = \ {\cdot}457 = \log \text{ of } \underline{2{\cdot}865}.$$
$$\log 19{\cdot}3 = 1{\cdot}286,$$
$$\tfrac{1}{5}\log 19{\cdot}3 = \ {\cdot}257 = \log \text{ of } \underline{1{\cdot}81}.$$
$$\log 1{\cdot}93 = 0{\cdot}286,$$
$$\tfrac{1}{5}\log 1{\cdot}93 = \ {\cdot}057 = \log \text{ of } \underline{1{\cdot}14}.$$

But now suppose we required the root of a fraction, *i.e.* of something whose logarithm was negative. We must think how to proceed in that case. Suppose for instance we want the fifth root of ·193,

$$\log {\cdot}193 = \bar{1}{\cdot}286,$$

that is to say a negative part and a positive part; it means

$$-1 + {\cdot}286.$$

In order to divide this by 5 conveniently, it is best to increase both the negative and the positive parts by any convenient equal amounts: in this case the convenient amount is 4. Add -4 to the negative part, and add $+4$ to the positive part: the value will thereby be unaltered, but it will be written as

$$-5 + 4{\cdot}286$$

and now it is quite easy to divide by 5, yielding

$$-1 + {\cdot}857 \quad \text{or} \quad \bar{1}{\cdot}857,$$

which is the log of ·72. Wherefore

$${\cdot}72 = \sqrt[5]{{\cdot}193}.$$

The root is bigger than the number. That is universal with roots of proper fractions. When we square a fraction we diminish it; when we square-root a fraction, consequently, we increase it. Think it out; it is all in accordance with common-sense.

But we must take another example.

Let us find the square root of ·0054.

$$\log \cdot 0054 = \bar{3}\cdot 732$$
$$= -4 + 1\cdot 732,$$
$$\tfrac{1}{2}\log \cdot 0054 = -2 + \cdot 866$$
$$= \bar{2}\cdot 866 = \log \text{ of } \cdot 0735,$$

wherefore $\cdot 0735 = \sqrt{\cdot 0054}$.

We might have made an approximate guess at this, because $\sqrt{\cdot 0049}$ could have been written down as ·07 by inspection, and so $\sqrt{\cdot 0054}$ will be a little bigger; how much bigger it is not so easy to guess.

But suppose we had wanted $\sqrt{\cdot 054}$, we should have found nothing like a 7 in the root. Let us do it:

$$\log \cdot 054 = \bar{2}\cdot 732,$$
$$\tfrac{1}{2}\log \cdot 054 = \bar{1}\cdot 366 = \log \text{ of } \underline{\cdot 232}.$$

So
$$\cdot 232 = \sqrt{\cdot 054}, \quad \text{whereas} \quad \cdot 0735 = \sqrt{\cdot 0054},$$
$$\cdot 0232 = \sqrt{\cdot 00054}, \qquad \cdot 735 = \sqrt{\cdot 54},$$
$$2\cdot 32 = \sqrt{5\cdot 4}, \qquad 7\cdot 35 = \sqrt{54},$$
$$23\cdot 2 = \sqrt{540}, \qquad 73\cdot 5 = \sqrt{5400}.$$

A little easy repetition on this point may be useful so as to emphasise it.

$$\sqrt{49} = 7, \text{ and } \sqrt{100} = 10;$$

so $\sqrt{4900} = 70,$

and $\sqrt{490000} = 700;$

also $\sqrt{\cdot 49} = \cdot 7,$

$$\sqrt{\cdot 0049} = \cdot 07,$$
$$\sqrt{\cdot 000049} = \cdot 007;$$

but if instead of two ciphers we had suffixed or prefixed only one cipher, we should have had quite different results, and not so easy to ascertain, viz. the following:

$$\sqrt{4\cdot9} = 2\cdot214,$$
$$\sqrt{490} = 22\cdot14,$$
$$\sqrt{49000} = 221\cdot4,$$
$$\sqrt{\cdot049} = \cdot2214,$$
$$\sqrt{\cdot00049} = \cdot02214.$$

Exercises.

$$\sqrt[3]{407} = 7\cdot41$$

because $\log 407 = 2\cdot610,$

and one-third of it is $\cdot870 = \log \text{ of } 7\cdot41.$

Similarly work out the following:

$$\sqrt[3]{\cdot407} = \cdot741,$$
$$\sqrt[3]{407000\cdot} = 74\cdot1,$$
$$\sqrt[3]{\cdot000407} = \cdot0741,$$

so that whereas for square roots the noughts can be added in pairs to leave the digits unaltered, for cube roots the ciphers must be added in triplets if they are to make no change in the digits. This is an immediate consequence of the fact that

$$\sqrt[3]{1000} = 10.$$

The last case, for instance, works out thus:

$$\log \cdot000407 = \bar{4}\cdot61 = -6 + 2\cdot61,$$

of which one-third is $\bar{2}\cdot870 = \log \text{ of } \cdot0741.$

The simplest way of dealing with these things however is to express them in powers of ten.

Thus $\cdot000407 = 407 \times 10^{-6},$

so its cube root is

$$\sqrt[3]{407} \times 10^{-2} = 7\cdot41 \div 100 = \cdot0741.$$

But now suppose the digits had not been added in triplets.

Find cube root of 40·7.

$$\log 40\cdot7 = 1\cdot61,$$

a third of that is ·537, which is the logarithm of 3·44... which is therefore the root required.

Again, to find $\sqrt[3]{4·07}$.

$$\log 4·07 = ·61,$$

one-third is 2033 and the number corresponding to this log is nearly 1·6.

So $(4070)^{\frac{1}{3}} = 16$ nearly; more accurately 15·966...

Also $$(·00407)^{\frac{1}{3}} = ·16.$$

Find the cube root of ·0078.

We may write it as $7·8 \times 10^{-3}$,

and so express its cube root as

$$1·98 \times 10^{-1} = ·198.$$

Find the cube root of ·000000081.

Express it as 81×10^{-9}.

Its cube root is $4·33 \times 10^{-3} = ·00433$.

Roots of negative numbers.

Perhaps it is not likely to occur often in elementary practice, but it is worth noticing that the cube root of a negative number is by no means impossible. What, for instance, is the cube root of -8; that is, what number multiplied twice by itself will make -8? The answer is -2, for

$$-2 \times -2 = +4 \quad \text{and} \quad +4 \times -2 = -8.$$

So $\sqrt[3]{-27} = -9$, $\sqrt[3]{-1728} = -12$, and so on.

Also $\sqrt[3]{-407} = -7·41$, see above.

The *square* root of a negative number has no simple meaning. If we tried to find the square root of -9 or -25 we could not do it, for $-3 \times -3 = +9$ and $-5 \times -5 = +25$. Hence negative numbers have no square roots, but they have cube roots. Having no square roots of course they cannot have fourth roots, for a fourth root is simply the square root of a square root. But they have fifth roots and seventh roots and any odd numbered roots, because an odd number of minus signs multiplied together make minus. Negative numbers have no even roots.

This is not all that can be said concerning the roots of negative numbers, by any means: Pure mathematicians know a great deal more than that about them; and later, children who like the subject may learn some of it, but not yet. In order however to prepare them for a convenient way of dealing with the matter, I will point out that any negative number can be said to have -1 as a factor; for instance,

$$-8 = 8 \times -1,$$
$$-16 = 16 \times -1,$$
$$-27 = 27 \times -1, \text{ and so on.}$$

Hence any root of any negative number is equal to the same root of the same positive number multiplied by the appropriate root of -1. For instance,

$$\sqrt[3]{-8} = \sqrt[3]{8} \times \sqrt[3]{-1} = 2\sqrt[3]{-1},$$
$$\sqrt[3]{-27} = \sqrt[3]{27} \times \sqrt[3]{-1} = 3\sqrt[3]{-1},$$
$$\sqrt[5]{-32} = 2\sqrt[5]{-1}, \text{ and so on.}$$

[Remember that $\sqrt{(xy)} = \sqrt{x}\sqrt{y}$ or that $(ab)^n = a^n b^n$.]

But the same method may be extended to even roots, thus

$$\sqrt{-16} = \sqrt{16} \times \sqrt{-1} = 4\sqrt{-1},$$
$$\sqrt{-9} = 3\sqrt{-1},$$
$$\sqrt[4]{-81} = 3\sqrt[4]{-1},$$
$$\sqrt[6]{-64} = 2\sqrt[6]{-1}, \text{ and so on.}$$

It is true that we do not yet know what to make of $\sqrt{-1}$ or $\sqrt[4]{-1}$ or $\sqrt[6]{-1}$; it is an impossible or imaginary quantity; but though we think that we do know what to make of $\sqrt[3]{-1}$ or $\sqrt[5]{-1}$, viz. although we know that they $= -1$, do not let us be too sure that we know all about even these. It is at any rate true that $-1 \times -1 \times -1 = -1$, and that is all that need now concern us; but it is not, strange to say, the *whole* truth concerning even the odd roots of minus one.

CHAPTER XXXI.

Geometrical Illustration of Powers and Roots.

GEOMETRICAL illustration, or illustration of number by simple diagrams, cannot be pressed very far with advantage for elementary purposes. But for simple things the illustrations are so vivid and useful and interesting that they should often be employed, and especially be set as exercises so as to infuse life and interest into what might otherwise be dull.

The simplest illustration of all relates to the squares or cubes of integers. That the square of 3 is 9 is illustrated in the most conspicuous manner by the diagram.

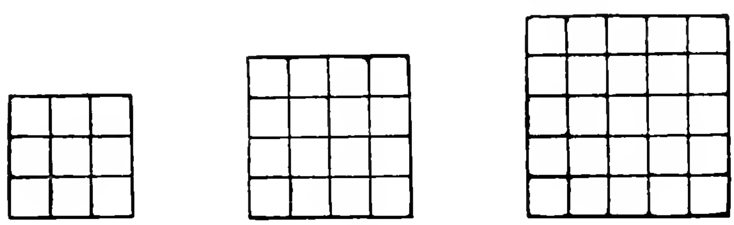


FIG. 12.

So also that the square of 4 is 16, and the square of 5 is 25.

That the cube of 2 is 8 is illustrated thus, but the best plan of dealing with solids is to use cubical wooden blocks and build them up.

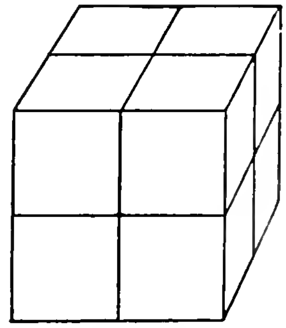


FIG. 13.

8 blocks will build a cube whose side is 2				
27	„	„	„	3
64	„	„	„	4

and so on.

The same blocks laid flat on the table will serve conveniently for squares and rectangles and commensurable areas generally. They will also serve to outline commensurable triangles: with conspicuous advantage in some cases.

By this kind of practice a reality about square and cube numbers is attained which can be got in no other way.

Naturally also the area of any rectangle can be thus illustrated as the product of length and breadth; and the volume of rectangular solids as the product of length, breadth, and height.

If we try to illustrate fourth or higher powers in this way we shall find ourselves helpless. Space is only of 3 dimensions. There are length and breadth and thickness, and no more. Some have tried to imagine what a fourth dimension would be like, but for the present we will be content with an actually experienced and familiar three dimensions.

So much for powers; now what about roots?

The few commensurable roots that exist must all be whole numbers, and they will be represented, so far as square and cube roots are concerned, by the length of the sides or edges of the squares or cubes which have so far been drawn or built up. Thus, for instance, the square root of 16 is 4, and the cube root of 27 is 3. But this fact, which is experimentally obvious in the commensurable case, where the square or the cube can be built of blocks, is true also in the general case. The length of a side of a square is the square root of its area always, and the length of the edge of a cube is the cube root of its volume always. This represents the geometrical notion of a root so far as geometry can illustrate it.

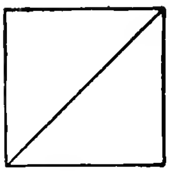


Fig. 14.

We will now proceed a little further.

Suppose we take a square and draw a diagonal across it, what is the length of that diagonal? It is evidently greater than a side, and not so great as two sides.

If we measure it carefully we shall find it rather less than a side and a half. It will be about one and two-fifths or 1·4 times a side.

Now construct a square on the diagonal, *i.e.* a fresh square with the old diagonal for one of its sides. We may not know how to do it accurately on blank paper, but it is quite easy to do if we use paper ruled faintly in squares, such as can easily be obtained in copy-books. Or the figure may be constructed by folding over the tongue of a sort of square envelope. In any case it is quite easy to see that the square on the hypothenuse is twice the area of the square on either side of the isosceles rt.-angled triangle. For produce the sides along the dotted lines.

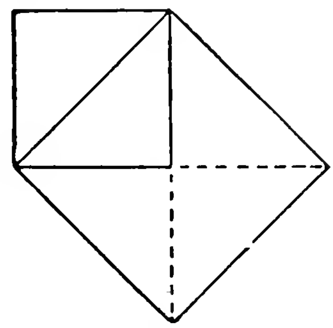


Fig. 15.

The larger square is thereby cut up into four parts each of which is half of the smaller square: see fig. 15. Therefore the areas of the squares are as 2 to 1.

But the side of a square is the square root of its area, hence a side of the new square is $\sqrt{2}$ times a side of the old one. In other words, the diagonal of a square is $\sqrt{2}$ times the length of one of the sides.

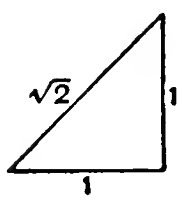


Fig. 16.

Or, expressing it otherwise, the hypothenuse of an isosceles right-angled triangle is $\sqrt{2}$ times either of the sides.

If we were to draw a square on one of the sides and a square on the hypothenuse, the two squares would be as 2 to 1.

(The area of the triangle itself is evidently $\frac{1}{2}$ on the same scale.)

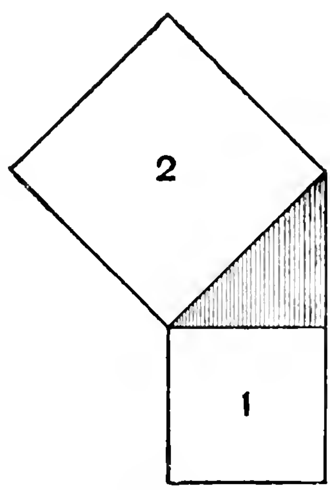


Fig. 17.

Drawn thus, we might not see how to prove it, but drawn as in the previous figure the proof is obvious. To be sure that there is no mistiness about it, a beginner should write the proof out for himself, expressing it as well as he possibly can. The inventing and writing out of proofs is good exercise, and to do it really well demands some thought and a little skill. The skill so cultivated is of a useful kind in life.

An example is necessary; but the danger of an example is that it is apt to become stereotyped. It may be varied in innumerable ways, and a way invented by the pupil is better than one which he has to learn. If there are actual errors in his proof they can be pointed out, but defect of taste and style, though much to be deprecated in adult persons, must be eliminated gradually from a beginner. He cannot be expected to concoct a proof in finished style from the first.

Something like the following would be good enough:—
To prove that the square on the hypothenuse of an isosceles right-angled triangle is double of the square on either of the sides.

Construction.—Draw the triangle ABC with right-angle at C, so that AC, BC are the equal sides, and AB the hypothenuse.

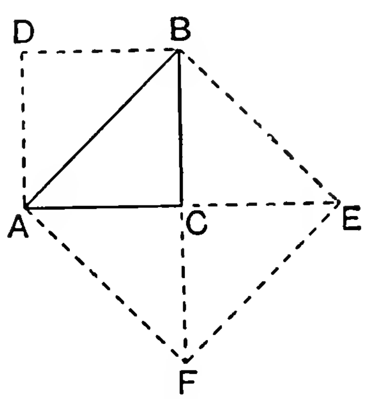


FIG. 18.

Now draw a square on AC, and draw it so that the equal sides of the triangle shall serve as two of the equal sides of the square. That is draw the square $ACBD$.

Next draw a square on AB, and draw it so that C lies in the middle of it, which is best done by producing AC an equal length to E, and producing BC an equal length to F, and then joining up so as to make the square $ABEF$, which, being a quadrilateral figure with equal diagonals at right angles to each other, must be a square.

Proof.—The square so constructed contains the area of the original triangle four times, while the former square contains it only twice. Therefore the square on the hypothenuse is double the square on one of the equal sides of an isosceles right-angled triangle. Q.E.D.

Beginners can and should realise the fact, immediately and without words, by having given to them small triangles in wood, and by then piecing them together so as to make the above figure. In a short time, left to themselves, the realisation becomes vivid.

Now proceed to a right-angled triangle with unequal sides. Suppose as a special case the hypothenuse is double one of the sides. It is not difficult to devise a way of drawing this figure if we use a pair of compasses.

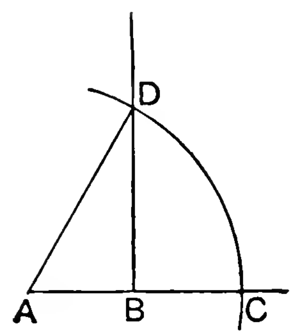


FIG. 19.

For let AB be one of the sides. Double it and you get AC. With centre A and radius AC mark off a circle. This gives the length of the hypothenuse.

At B draw a line perpendicular to AB till it meets the circle at D; then join A and D. The triangle ABD is the triangle required, viz. a right-angled triangle with its hypothenuse double one of the sides.

If we were to draw a square on AD and another on AB, the area of the one square would be quadruple that of the other; because the sides are as 2 to 1, therefore the squares of the sides will be as 4 to 1.

What about a square drawn on BD? If drawn and measured it will be found to be about $\frac{3}{4}$ of the big square. It can be shown by geometry that its area is *exactly* three quarters of the big one. In other words, that the middle sized square and the small one added together exactly equal the big square in area. This is a most curious and important fact: about as important as anything we have come across, if it applies, as it does, to all plane right-angled triangles without exception. But they must be *plane* triangles; *i.e.* the sides must be straight.

We are not yet supposed to know how to prove it. We can verify it approximately by drawing the squares carefully and cutting them out in wood or cardboard or sheet lead, and then weighing them. The two smaller squares will be found just to balance the big one, if they are cut out neatly and if the sheet was uniform in thickness and material. This is not a proof that they are mathematically equal, but it is a verification that they are approximately equal, equal "within the limits of error of experiment."

That is a kind of equality by no means to be despised. In some difficult cases it is all the equality that can be ascertained. In the present case it is by no means all; but no proof of exact equality can be obtained by empirical or experimental processes, no matter how carefully they are carrried out. Exactness is a prerogative of mathematical reasoning, that is reasoning on pure abstractions from which all flaws and imperfections and approximations are by hypothesis eliminated.

The fact that the squares on the sides of any right-angled triangle are together equal to the square on the hypothenuse, was known to the ancients. It was called the theorem of Pythagoras; and a classical proof, a fine example of ingenious reasoning, is given as the 47th proposition of the collection of geometrical propositions made by the Greek Geometer Euclid in his first book. Translations of that ancient treatise are sometimes still learnt by schoolboys in this country, and may be considered a part of classical education. It is an antiquated and slow way of learning geometry however, and in fact can hardly be intended seriously for that purpose. Nevertheless it is a delightful literary work and pleasant for reference.

People who are not acquainted with it are hardly educated in the usual English sense.

Many proofs can be given of any proposition, and the fact itself is of more importance than any one proof of it.

That does not for a moment mean that proofs can be dispensed with, for without a proof we should not really know the fact. We could know it approximately but not rigorously and exactly; and it should be always a joy to feel that a theorem or a statement can be made without limitation or approximation. Such statements are the only ones that can be pressed into extreme cases, with perfect confidence that whenever applicable, that is whenever the postulated data are satisfied, they will be always precisely true.

What we are doing at present however does not necessarily demand extreme accuracy. We have been finding roots, which we can only do approximately, and we now want to illustrate them. It will suffice for our present purpose if we assume Pythagoras's theorem as experimentally verifiable with sufficient accuracy for our present purpose, and proceed to use it.

The most remarkable of all right-angled triangles is the one whose sides are all commensurable, namely 3, 4 and 5. The square of 5 is equal to the sum of the squares of 4 and 3.

$25 = 16 + 9.$ See fig. 20.

Of course the sides might equally well be 6, 8, and 10;
also 9, 12, and 15;
12, 16, and 20;
and so on.

Also they could be 1·5, 2, 2·5;
·75, 1, 1·25;
and so on.

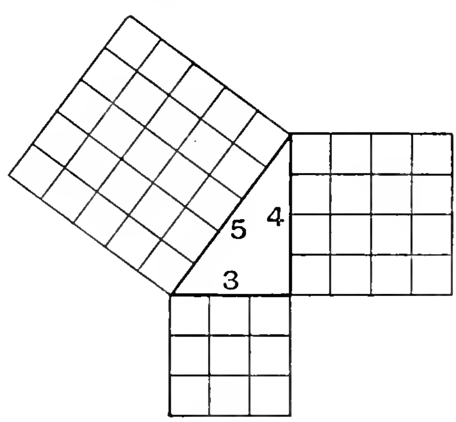


FIG. 20.

So long as the *proportion* holds, the absolute length of the sides is only a matter of "scale."

There is no other commensurably-sided right-angled triangle until we come to the one with sides 5, 12, 13; and the next one has sides 8, 15, 17. [See Appendix.]

Triangles with commensurable sides can be outlined by children by surrounding them with square blocks or slabs; and it is especially instructive to outline right-angled triangles in this way, because then the squares on the three sides can, after suggestion, be completed, and the number of blocks in each counted: when it will be perceived that

$$9+16 = 25, \quad 144+25=169, \quad 225+64 = 289;$$

a fact which ought to arouse some curiosity, since it represents the first inkling of one of the most simple fundamental and universal truths in existence.

What we have learnt by assuming Pythagoras's proposition, so far, enables us to say that in a right-angled triangle with the hypothenuse double the base the vertical side is $\sqrt{3}$ times the base. For the squares on each are as $1:3:4$; therefore the sides are as $\sqrt{1}:\sqrt{3}:\sqrt{4}$, that is as $1:\sqrt{3}:2$.

If the hypothenuse is treble the base, the squares will be as 9 to 1, and so the square on the vertical side will be represented by 8 on the same scale, and the vertical side itself will be $\sqrt{8}$, which equals $2\sqrt{2}$.

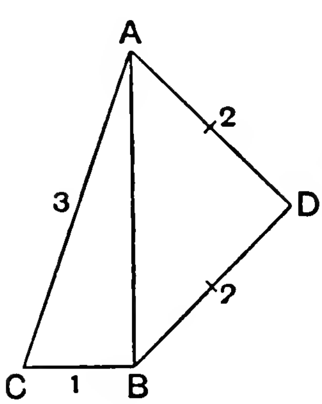


FIG. 21.

This should be examined and verified.

It will be easy for a beginner to devise a verification of it. For instance, thus:

Draw any vertical AB. Draw half a square on it, as shown, ADB.

Take half one of these sides, and lay it off horizontally, BC.

Then, this being called 1, BD or AD will be 2, and AB will be $2\sqrt{2}$ or $\sqrt{8}$; and so therefore AC should equal 3 on the same scale, because $8+1 = 3^2$. See if AC does measure 3.

Further geometrical methods of finding square roots.

Let us now attend a little more carefully to the important statement that the square root of the area of a square is the length of one of the sides. We have seen that it is true numerically, now see if it is true and sensible physically. The point to attend to is that the square of a length is an area, and the square root of an area is a length, not proportional to a length or numerically represented by a length, but actually and physically a length.

$$\sqrt{a^2} = a.$$

a being a length, a^2 is an area, and $\sqrt{a^2}$ is therefore a length again. But there is no reason why the area need be square. Suppose it were oblong, and given as $a \times b$; if $\sqrt{(ab)} = x$, x would still be a length. What length?

Answer. The length whose square is equal to the product ab, the geometric mean of the two lengths a and b; for if

$$\sqrt{ab} = c,$$

$$ab = c^2,$$

and so $$\frac{a}{c} = \frac{c}{b} \quad \text{or} \quad a : c = c : b,$$

or the three quantities a, c, b are in geometrical progression, for they differ by a constant factor, viz.

$$r = \frac{c}{a} = \frac{b}{c}.$$

They might be written $\frac{c}{r}$, c, cr.

The term c is the mean of the other two terms in the G.P., so it is called their geometric mean.

Can it be found geometrically? It can, and this is another most interesting proposition known to the ancients and recorded by Euclid. It is called the 14th proposition of his second book. Though perhaps not easy to prove, it is extremely easy to state. We will state it now and prove it later. The statement without a proof is a poor thing, but the statement *as a prelude to the proof*—a statement which shall provide a niche for the proof in the mind of a beginner and cause him to welcome it when it comes—is an excellent thing.

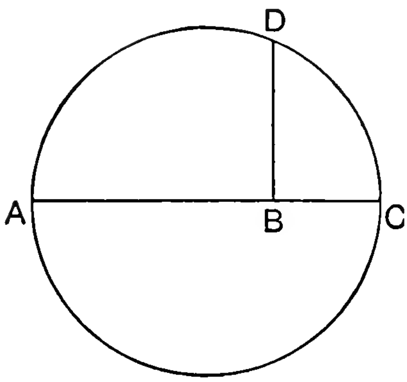


FIG. 22.

Construction for finding the geometric mean of two lengths.

Lay off the lengths end to end as $AB+BC$.

Draw a circle on the combined lengths as diameter, and erect a perpendicular at the junction-point B till it meets the circle in D; then BD is the geometric mean of AB and BC.

The figure shall be repeated below, with the lengths labelled, and the rectangle ab shown. (Fig. 23).

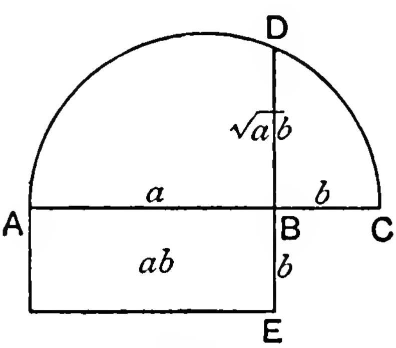


FIG. 23.

"Geometric mean" is an arithmetical or algebraical sort of term. What will it mean geometrically? It will mean that the square on BD has to equal in area the so-called rectangle $AB.BC$, which means the real rectangle $AB.BE$. That is to say the square on the length $\sqrt{ab}$ has to equal the area ab.

That is precisely what remains to be proved.

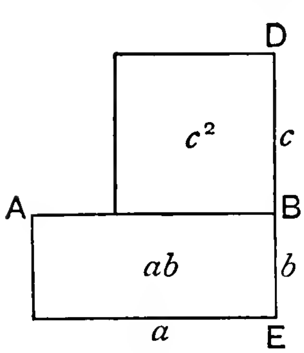


FIG. 24.

If $$c^2 = ab$$
then c is the geometric mean of a and b,

$$\text{or} \quad \frac{a}{c} = \frac{c}{b}.$$

The only practical difficulty is how to find the length c, and that is overcome in a very simple manner by the circle in the above construction.

If any one has got as far as the 35th proposition of Euclid's third book, they can devise a proof of this curious and very important property of the circle for themselves; in fact the figure annexed suggests it at once, as soon as we know that the rectangles contained by the segments of two chords are equal.

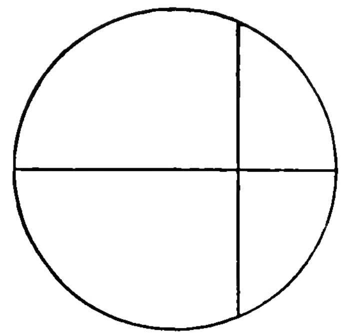


FIG. 25.

Given this simple and beautiful construction, we can at once find a length numerically representing the square root of any given number n; for we can take the two initially given lengths as n and 1 respectively, so that their product is n, and the geometric mean will then represent the square root of n, because it will be equal to $\sqrt{(n \times 1)}$. (Fig. 26).

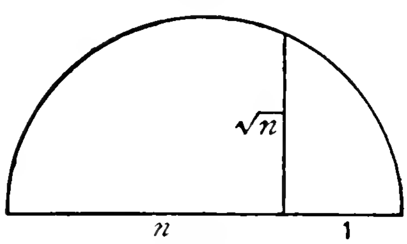


FIG. 26.

For instance to construct $\sqrt{4}$.

Take a line 5 inches long as the diameter of the circle, mark off 4 inches and draw a perpendicular to meet the circle; this will be $\sqrt{4}$, and if measured will be found to equal 2 inches.

To find $\sqrt{5}$ geometrically.

Draw a circle of radius 3 inches, so that its diameter is 6 inches. At the first inch draw a perpendicular and measure its length. That will be the root of 5. It should equal 2·236 inches if carefully drawn and measured.

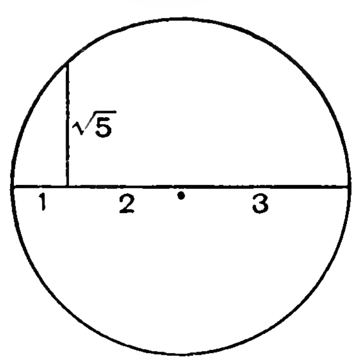


FIG. 27.

For the root of 7 the same construction exactly is to be carried out, only the circle will be 4 inches in radius.

For the root of 2 the circle will be 1·5 inches radius, or 3 inches diameter.

And for the root of any number whatever, n, the radius of the circle will be $\frac{n+1}{2}$.

For roots of large numbers, this method will not be convenient, but for roots of fractions not too far removed from unity it serves well.

For instance, to find the root of 3·6. Take a circle of 2·3 inches radius, and make the construction, erecting a perpendicular to the diameter at the end of the first inch. Its length gives the root, and should equal 1·9 inches.

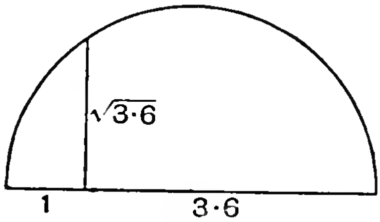


FIG. 28.

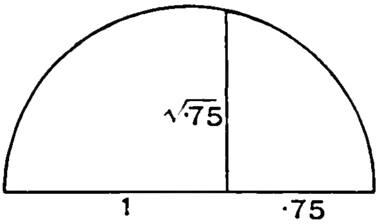


FIG. 29.

To find the root of ·75.

Take a circle $\frac{1}{2} \times 1{\cdot}75 = {\cdot}875$ inch radius, and at the first inch of it erect the perpendicular.

Its length will be greater than ·75, as necessary for the root of a proper fraction, and it should equal ·866.

This particular result could however have been still more easily calculated, or at least expressed in terms of $\sqrt{3}$; for ${\cdot}75 = \frac{3}{4}$.

so $$\sqrt{{\cdot}75} = \frac{\sqrt{3}}{\sqrt{4}} = \tfrac{1}{2}\sqrt{3} = \tfrac{1}{2} \times 1{\cdot}732 \ldots = {\cdot}866 \ldots.$$

It must be understood then that a geometrical construction in these cases, though it may be regarded as a simple method of arriving at the result, is more particularly an *illustration* of a result otherwise arrived at. This is however not always the case, and sometimes by construction results can be found which it would be extremely difficult to get in any other way. Engineers and building constructors know this well: and graphical methods are in constant practical use.

CHAPTER XXXII.

Arithmetical method of finding Square Roots.

We now know three methods of finding a square root.

1. The factor method, when it is applicable, which it seldom is; whenever it is easily applicable it should be used. Often it becomes a matter of guessing and trial and error, with the error gradually corrected or diminished.

2. The logarithm method, which is the real practical plan, and is frequently done with a slide rule.

3. The graphical method.

4. There is however another, an arithmetical method, which is usually learnt, though seldom really employed. It is an ingenious plan and is not at all bad for finding square roots. For cube roots it gets complicated, and for higher roots like fifth and seventh it would be altogether too difficult for anyone but a mathematician, and he would never think of employing it.

To find a really high root, for instance a 9th root, the logarithm method is the only reasonable one; though we might take the cube root twice over. A sixth root is the square root of a cube root. An eighth root is the result of a square root operation three times repeated. An eleventh root I could only do by logarithms, and with them it is so easy that nothing better is needed. Let us see, for instance, what is the eleventh root of 2,

$$\frac{\cdot 30103 \ldots}{11} = \cdot 0273664 \ldots = \log \text{ of } 1{\cdot}06503 \ldots,$$

which is therefore the root required.

[If any part of such an answer as the above pretended to "circulate," we should know that the recurrence was spurious, and only due to the fact that not enough digits in log 2 had been taken into account. Roughly speaking we may say that all numbers are incommensurable, except those specially selected to be otherwise.]

Why then learn any arithmetical method for finding square roots, other than the logarithm method?

Answer. Because we might not have a table of logarithms handy, and because it is ignominious to be dependent on material tools except in operations which are complicated.

To find a cube root by direct process is rather complicated, and I do not recommend its being learnt except by enthusiasts: and they will forget it again. But the rule for square root is fairly easy and often useful. It will however be the hardest thing we have attempted yet, and the proof will be deferred to the next chapter. It is not usually considered hard, but all the things before this have been easier in reality, though people often shy at them. I hope they will do so no longer.

To find the square root of 256 by direct arithmetic. Set it down like a long division sum, but with the digits marked out in pairs, by dots or commas or other marks, as shown, beginning with the units place, then work as follows:

$$\begin{array}{r@{\,)\,}l}
1 & \dot{2}5\dot{6}\ (\ 16 \\
 & \underline{1} \\
26 & 156 \\
 & \underline{156} \\
\end{array}$$

First guess the square root of 2, or the integer smaller.

It is 1, so put it in two places, and multiply and subtract as in long division. Then double the 1, and place it on the left as 2, and see how many times it will go into something less than 15; guess 6.

Set down 6 in two places as shown, multiply and subtract, and there is no remainder. The sum comes to an end: the root is 16.

If we had guessed 7 instead of 6, as might seem natural, the product treated as above would have been 189, and been too big.

If we had been given the number 2560, it would have been dotted off in pairs as follows:

$$2\dot{5}6\dot{0},$$

and the result would have been quite different. We should now have to guess the root, not of 2, but of 25, which is very easily done. The process would then have looked like this:

```
   5 ) 2560 ( 50·6
       25
       ------
1006 )   6000
         6036
         ----
         - 36
```

so that 50·6 is approximately the square root of 2560.

The small remainder shows that the result is not quite accurate, and its negative value shows that the result is slightly in excess.

(Observe that ciphers, like the other figures, are always brought down in pairs. If it were a cube root we were finding they would be brought down in triplets.)

It is natural to put 6 in the second stage, after the 0, as we have done above, because it is very nearly right. It is a little too big however, and if we wanted to work the root out further, we should put 5 and be sure that the next figure would be 9.

A more exact result is 50·5964426....

To find the square root of 6241.

Set it down, and again partition off the figures in pairs, beginning with the units place, by dots or other marks, as shown

```
      7 ) 6̇24̇1 ( 79
          49
        ------
    149 ) 1341
          1341
          ----
          ....
```

Guess the root of 62 or the next lower integer; guess 7. Set it down in 2 places, multiply, and subtract. Double 7, and see how many times it will go in 134; guess 9 times.

Set down 9 in 2 places, multiply as shown, and subtract.

There is no remainder: the root required is 79 exactly.

We might have guessed this. Looking at the number we see that the root will be less than 80, for $80^2 = 6400$. But it will not be much less than 80, because a moderate difference in a square is but a small difference in the root. So we might try 78. Multiplying out, we should find $78 \times 78 = 6084$, which is about as much too small as the other was too big. Hence we know that it is either 79 or something very near to 79.

Take another instance of guessing: choosing a number quite at random, say $\sqrt{(596)}$. We know that

$$24^2 = 4 \times 12^2 = 4 \times 144 = 576,$$

while $25^2 = 625$. So here again the number lies about half way between 24 and 25, but a little nearer the smaller of the two; and we might see how 24·4 would answer.

Multiplying $24{\cdot}4 \times 24{\cdot}4$ we should get 595·36, which is very close. As a matter of fact,

$$\sqrt{(596)} = 24{\cdot}4131112\ldots$$

which you can proceed to ascertain by the arithmetical process worked out at length, as thus:

```
        2 | 596 | 24·41311123
          | 4
       44 | 196
          | 176
      484 | 2000
          | 1936
     4881 | 6400
          | 4881
    48823 | 151900
          | 146469
   488261 | 543100
          | 488261
  4882621 | 5483900
          | 4882621
 48826221 | 60127900
          | 48826221
488262222 | 1130167900
          |  976524444
          |  153643456
```

We see that the next digit will be 3, and have placed it in position, but we consider that as we have now obtained ten significant figures, we have gone far enough, especially as we know that there can be no end.

If we have to find the square root of a decimal, we can mark it off into pairs, as before, always beginning with the units place. Thus $1\dot{7}{\cdot}8\dot{5}3\dot{4}$ is marked off properly for the purpose of extracting its square root, which is plainly 4 decimal something.

So also $\dot{0}{\cdot}0\dot{0}05\dot{7}6$ is properly marked off, and its root is ·024.

The marking off in pairs is manifestly connected with the fact that $\sqrt{100} = 10$. It is to get the power of ten in the answer right. The number of dots gives the number of figures in the answer, if the units place is included in it. To find a cube root, the dots would be placed on every third digit, but always beginning with the units place, because any root of 1 is 1.

There is not much more than this to be learnt about this ingenious and practical process, until we are able to prove it and see the reason of the successive steps: this will be fully attended to in the next chapter, pages 296 to 299. There are however a great number of far more important things, and I only place this brief record of the process here, because I by no means wish to extrude it; moreover it is an interesting thing to prove. It is essentially a limited process, however, since, for any useful purpose, it only applies to *square* roots; though a complication of it, on the same principle, will apply to cube, and even to higher, roots. At the same time it is undeniable that square roots and cube roots occur much more frequently than do others, just as second and third powers do; partly because they cover the actual dimensions of our space.

CHAPTER XXXIII.

Simple Algebraic Aids to Arithmetic, etc.

A VERY little knowledge of algebra enables us to make better estimates, and to approximate as closely as we please, both to powers and to roots; and it is worth while to show this now: this chapter being chiefly one for exercise and practice. It may be regarded as a chapter of miscellaneous worked out examples, rather than as a progressive chapter; though it contains the proof or explanation of the ordinary square root rule.

First of all consider the multiplication of two binomials, that is two factors each consisting of two terms, say $(a+b)(c+d)$. Every term will have to be multiplied by every other, for it means $a(c+d)+b(c+d)$, that is $ac+ad+bc+bd$.

So for instance $(3+\sqrt{2})(4+\sqrt{3})$

will equal $(3\times 4)+3\sqrt{3}+4\sqrt{2}+\sqrt{2}\sqrt{3}$

$$=12+3\sqrt{3}+4\sqrt{2}+\sqrt{6}.$$

Or take this example,

$$(\sqrt{2}+2)(\sqrt{3}-3)$$

multiplied out it becomes $\sqrt{6}+2\sqrt{3}-3\sqrt{2}-6$.

But take a more easily verifiable example, say

$$(17-5)(13-10)$$
$$=221-170-65+50$$
$$=271-235=36;$$

rather an absurd way to do such simple arithmetic as 12×3.

Very well, now take the case of $(a+b)^2$.

It means $(a+b)(a+b)$.

And this multiplied out equals $a^2+ab+ba+b^2$;

but $ab+ba = 2ab$,

so $$\mathbf{(a+b)^2 = a^2+2ab+b^2.}$$

Similarly $$\mathbf{(a-b)^2 = a^2-2ab+b^2.}$$

Now let us use these results to obtain powers and to approximate to roots. Suppose we want 103^2, work it out thus:

$$\begin{aligned}(103)^2 &= (100+3)^2\\ &= 100^2+6\times 100+3^2\\ &= 10{,}000+600+9\\ &= 10609.\end{aligned}$$

Again, to find $(998)^2$; write it as $(1000-2)^2$

$$\begin{aligned}&= (1000)^2-4000+4\\ &= \text{one million less } 3996\\ &= 996004.\end{aligned}$$

Similarly: $$(125)^2 = (120+5)^2 = 14400+1200+25 = 15625$$

or $$(125)^2 = (130-5)^2 = 16900-1300+25 = 15625.$$

$$(79{\cdot}2)^2 = (80-{\cdot}8)^2 = 6400-1{\cdot}6\times 80+{\cdot}64 = 6400{\cdot}64-128 = 6272{\cdot}64.$$

$$(5{\cdot}11)^2 = (5+{\cdot}11)^2 = 25+1{\cdot}1+{\cdot}0121 = 26{\cdot}1121.$$

$$(39)^2 = (40-1)^2 = 1600-80+1 = 1521.$$

A further algebraical aid is often of great use, especially in preparing for logarithmic calculations.

The value of $(a+b)(a-b)$ when multiplied out is

$$a^2-ab+ba-b^2;$$

the two middle terms destroy each other, and so only $a^2 - b^2$ is left.

This is a most useful fact to remember,

$$\mathbf{a^2 - b^2 = (a+b)(a-b).}$$

For instance $9^2 - 4^2 = (9+4)(9-4) = 13 \times 5 = 65$,

$$(17{\cdot}31)^2 - (2{\cdot}69)^2 = 20 \times 14{\cdot}62 = 292{\cdot}4,$$
$$({\cdot}019)^2 - ({\cdot}008)^2 = {\cdot}027 \times {\cdot}011 = 2{\cdot}97 \times 10^{-4},$$
$$(1{\cdot}05)^2 - ({\cdot}95)^2 = 2 \times {\cdot}1 = {\cdot}2.$$

The fact is so important that it is worth learning in words.

The difference of two squares is equal to the product of sum and difference.

Expressed thus it suggests a geometrical way of putting it:

Let AB and AC be any two given lengths.

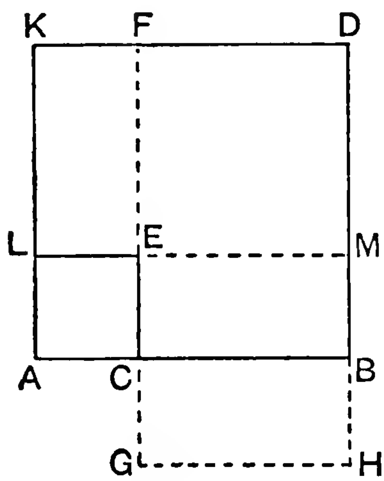


Fig. 30.

Erect a square on each, viz. the square AD and the square AE, drawing them so that they are superposed.

The difference of the two squares is shown by the irregular six-sided rectangular figure with what is called a "re-entrant" angle at E.

We have to show that this area is equal to that of a rectangle bounded by lengths representing the sum and the difference respectively of the two given lengths.

To construct such a rectangle in a convenient position, produce CE both ways to F and G, making $CG = CA$. Then FG is equal to the sum of the two given lengths, viz. $AB + AC$; and GH, which is the same as CB, is equal to their difference.

Therefore the area of the rectangle $GHDF$ exhibits the product of the sum and difference. Hence we have to show that this rectangle is equal to the area of the irregular figure $CELKDBC$, the difference of the two squares.

Now the two areas have a great part common, viz. the rectangle BF; so we have only to show that the residues LF and GB are equal.

By producing LE to M, another rectangle EB is constructed equal to GB; and this rectangle is plainly equal to LF, because the height and base of the one correspond to the base and height of the other.

The proof is therefore completely indicated. It has been rather long and not particularly neat, but it is such a proof as could be invented by an industrious beginner for himself. The proposition is really an ancient one, and is established with due ceremony in Euclid Book II., Propositions 5 and 6.

We observe from this example that a geometrical proof is or may be hard, while an algebraic proof of the same thing is absurdly easy: so it often is, though not always. As usual there are plenty of ways of proving a proposition; the proposition itself is more important than any one proof of it.

The geometrical illustration has been introduced here to emphasise the extreme importance and usefulness of the fact that

$$(x+y)(x-y) = x^2 - y^2.$$

Now let us proceed to show how it is employed for adapting things to logarithmic calculation.

Suppose we had to find the value of the following:

$$(8{\cdot}131)^2 - (4{\cdot}026)^2.$$

We might look out the logarithm of each, double it, find the antilog of each, and then subtract them.

But on the other hand we might first throw it into the form

$$12{\cdot}157 \times 4{\cdot}105,$$

look out the logarithms of these two numbers, add them, and find the antilog of the sum. And this is a shorter process than the preceding.

In general, sums and differences are awkward for logarithmic calculation, while products and quotients are convenient.

Take another example of finding the value of a difference of two squares:

$$\left(\frac{1}{15}\right)^2 - \left(\frac{1}{35}\right)^2 = \left(\frac{1}{15}+\frac{1}{35}\right)\left(\frac{1}{15}-\frac{1}{35}\right)$$

$$= \frac{7+3}{105} \times \frac{7-3}{105}$$

$$= \frac{40}{(105)^2}.$$

And it is easy to look out the necessary logarithms:

$$\log 40 = 1\cdot 6021 ; \quad \log 105 = 2\cdot 0212.$$

$$2 \log 105 = 4\cdot 0424$$

$$\text{difference } \bar{3}\cdot 5597 = \log \text{ of } \cdot 003625.$$

·003625 is therefore the result.

We might indeed have done the above differently, because we happen to see a common factor in the given expressions, and can take it outside brackets, thus,

$$\left(\frac{1}{15}\right)^2 - \left(\frac{1}{35}\right)^2 = \left(\frac{1}{5}\right)^2\left\{\left(\frac{1}{3}\right)^2 - \left(\frac{1}{7}\right)^2\right\} = \frac{1}{25}\cdot\frac{7-3}{21}\cdot\frac{7+3}{21}$$

$$= \frac{40}{25\times(21)^2} = \frac{160}{44100},$$

$$\log 16 = 1\cdot 2041$$

$$\log 4410 = 3\cdot 6444$$

$$\text{difference } \bar{3}\cdot 5597 = \log \text{ of } \cdot 003625 \text{ as before.}$$

This therefore serves as a check, and is itself instructive.

Sums of this kind, given as exercises, will call out nascent ingenuity and will furnish much better and more real arithmetical practice than a quantity of routine examples without much variety.

In so far as the actual arithmetical operations to be performed are usually simple and short, that is a peculiarity characteristic of nearly all the real sums that have to be done in practice; always excepting the long and intricate operations occasionally undertaken for special purposes by pure mathematicians—a matter with which children have nothing whatever to do.

Sometimes the converse use of the proposition

$$a^2 - b^2 = (a+b)(a-b)$$

is convenient. For instance, suppose we had to find the value of

$$(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}).$$

It would be very clumsy to interpret it arithmetically thus:

$$(1{\cdot}732 + 1{\cdot}4142)(1{\cdot}732 - 1{\cdot}4142)$$
$$= 3{\cdot}1462 \times 0{\cdot}3178,$$

whose logarithm is $\cdot 4977$
plus $\bar{1}{\cdot}5022$
equals $\bar{1}{\cdot}9999$

which is the log of something extremely near to unity, and perhaps unity itself if we had taken more places in the logarithms.

I say this would be an extremely clumsy way.

The neat and direct way is to write the product as the difference of the two squares, thus:

$$(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = (\sqrt{3})^2 - (\sqrt{2})^2 = 3 - 2 = 1,$$

which shows that it *is* unity exactly.

Take other examples of $(a+b)(a-b) = a^2 - b^2$:

$$(\sqrt{14}-\sqrt{8})(\sqrt{14}+\sqrt{8}) = 14 - 8 = 6.$$
$$(\sqrt{7}-\sqrt{3})(\sqrt{7}+\sqrt{3}) = 7 - 3 = 4.$$
$$(\sqrt{5}+1)(\sqrt{5}-1) = 5 - 1 = 4.$$
$$(\sqrt{57}-1)(\sqrt{57}+1) = 57 - 1 = 56.$$
$$(1+\sqrt{17})(1-\sqrt{17}) = 1 - 17 = -16.$$
$$(\sqrt{3{\cdot}14159}+1)(\sqrt{3{\cdot}14159}-1) = 2{\cdot}14159.$$

$$\{1+(\cdot 0012)^{\frac{1}{2}}\}\{1-(\cdot 0012)^{\frac{1}{2}}\} = 1 - \cdot 0012 = \cdot 9988.$$

$$\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right)\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right) = 3 - \tfrac{1}{3} = 2\tfrac{2}{3}.$$

This last might have been done thus:

$$\frac{3+1}{\sqrt{3}} \cdot \frac{3-1}{\sqrt{3}} = \frac{8}{3}.$$

$$(6+\sqrt{20})(6-\sqrt{20}) = 36 - 20 = 16.$$

$$(\sqrt{5}-2)(\sqrt{5}+2) = 5 - 4 = 1.$$

$$(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) = x - y.$$

$$(m+\sqrt{n})(m-\sqrt{n}) = m^2 - n.$$

$$(\sqrt{p}-q)(\sqrt{p}+q) = p - q^2.$$

$$\left(a-\frac{n}{b}\right)\left(a+\frac{n}{b}\right) = a^2 - \frac{n^2}{b^2} = \frac{a^2b^2-n^2}{b^2}.$$

$$\left(\frac{a}{b}+\frac{c}{d}\right)\left(\frac{a}{b}-\frac{c}{d}\right) = \frac{a^2}{b^2} - \frac{c^2}{d^2}.$$

$$\left(\frac{m}{\sqrt{x}}+\frac{n}{\sqrt{y}}\right)\left(\frac{m}{\sqrt{x}}-\frac{n}{\sqrt{y}}\right) = \frac{m^2}{x} - \frac{n^2}{y}.$$

$$\left(\frac{x}{\sqrt{a}}+\frac{a}{\sqrt{x}}\right)\left(\frac{x}{\sqrt{a}}-\frac{a}{\sqrt{x}}\right) = \frac{x^2}{a} - \frac{a^2}{x} = \frac{x^3-a^3}{ax}.$$

$$(30+\tfrac{1}{9}\sqrt{x})(30-\tfrac{1}{9}\sqrt{x}) = 900 - \frac{x}{81}.$$

$$\left(\frac{4}{x^2}-\frac{x^3}{6}\right)\left(\frac{4}{x^2}+\frac{x^3}{6}\right) = \frac{16}{x^4} - \frac{x^6}{36}$$

$$= \frac{576-x^{10}}{36x^4}.$$

$$(ab-\sqrt{ab})(ab+\sqrt{ab}) = a^2b^2 - ab.$$

$$(\sqrt[3]{a}+\sqrt[3]{b})(\sqrt[3]{a}-\sqrt[3]{b}) = a^{\frac{2}{3}} - b^{\frac{2}{3}}.$$

$$(a^x+b)(a^x-b) = a^{2x} - b^2.$$

$$(a^n+a^{-n})(a^n-a^{-n}) = a^{2n} - a^{-2n}$$

$$= a^{2n}(1-a^{-4n}).$$

$$(a^m + a^n)(a^m - a^n) = a^{2m} - a^{2n}.$$

$$(a\sqrt{x} + b\sqrt{y})(a\sqrt{x} - b\sqrt{y}) = a^2x - b^2y.$$

$$(1 + a^n)(1 - a^n) = 1 - a^{2n}.$$

$$\left(1 + \frac{1}{\sqrt{x}}\right)\left(1 - \frac{1}{\sqrt{x}}\right) = 1 - \frac{1}{x} = \frac{x-1}{x}.$$

$$\left(4\sqrt{x} + \frac{3}{\sqrt{x}}\right)\left(4\sqrt{x} - \frac{3}{\sqrt{x}}\right) = 16x - \frac{9}{x} = \frac{16x^2 - 9}{x}$$

$$= \frac{(4x+3)(4x-3)}{x}.$$

$$(1 + \sqrt{\log n})(1 - \sqrt{\log n}) = 1 - \log n.$$

$$(\sqrt{1\cdot 73y} + \sqrt{\cdot 73y})(\sqrt{1\cdot 73y} - \sqrt{\cdot 73y}) = 1\cdot 73y - \cdot 73y = y.$$

$$(\sqrt{(1+m)\,.\,u} - \sqrt{m\,.\,u})(\sqrt{(1+m)\,.\,u} + \sqrt{m\,.\,u}) = u^2.$$

$$(\sqrt{a+b} + \sqrt{b})(\sqrt{a+b} - \sqrt{b}) = a.$$

$$\left(1 + \sqrt{\frac{b}{a+b}}\right)\left(1 - \sqrt{\frac{b}{a+b}}\right) = 1 - \frac{b}{a+b} = \frac{a}{a+b}.$$

If we have now driven home the important fact that

$$(a+b)(a-b) = a^2 - b^2 \quad \ldots\ldots\ldots\ldots\ \ldots\ldots\ldots(1)$$

sufficiently, we will proceed to illustrate geometrically those other equally important truths, viz. that

$$(c+b)^2 = a^2 + 2ab + b^2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$(a-b)^2 = a^2 - 2ab + b^2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

or, expressed in words, **the square of a binomial is the sum of the squares of its terms plus twice their product.**

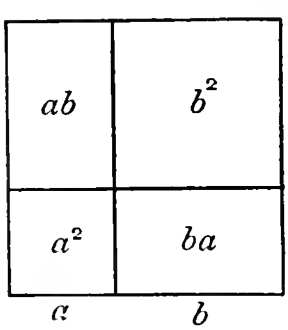


FIG. 31.

Or expressed geometrically. (2) The square on a line made up of two parts is the sum of the squares on the parts plus twice the rectangle contained by the parts.

The annexed figure makes this obvious. For the base of the big square is made up of two parts labelled a and b.

And we see that it is built up of the square on a, plus a square virtually on b, plus two rectangles each equal to the product ab.

It is a quite ancient proposition known as the fourth proposition of Euclid's second book.

(3) The statement for the squared difference $(a - b)^2$ expression may be worded geometrically thus:

If a straight line is divided into two parts, the sum of the squares on the whole line and one of the parts is equal to twice the rectangle contained by the whole line and that part together with the square on the other part.

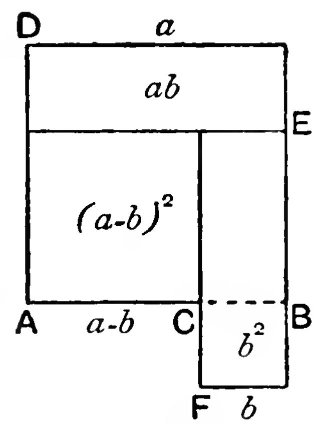


FIG. 32.

The same figure serves, differently labelled; but a separate figure may make it clearer. The square of AB, which is a^2, together with the square on BC, which is b^2, exceeds the square on AC, which is $(a - b)^2$, by twice the rectangle ab, that is by the two rectangles DE and EF.

This proposition, which asserts that

$$a^2 + b^2 = (a - b)^2 + 2ab,$$

is known as the 7th proposition of the second book of Euclid. It may be illustrated, like the preceding, by the folding of paper.

The process of putting these propositions into proved geometric form is, we see, liable to be rather troublesome and long. Algebraically they are quite easy. Geometry illustrates the algebra, but it does not in this latter instance illustrate it strikingly; and it is quite possible to spend too much time over such geometrical illustrations, unless they are made out by pupils for themselves, which is an admirable exercise. A great deal, though not all, of Euclid's second book is of this character, and represents an antique method of expressing algebraic results without employing algebra. For good reason in those days,—because algebra was not then invented. Children need not be dosed with too much of this rather confusing and nearly useless kind of geometry at the present time.

Illustrations.

Let us write down some illustrations of the use of these results in simplifying algebraic expressions, and in finding roots. Write the results compactly thus,

$$(a \pm b)^2 = a^2 \pm 2ab + b^2,$$

and then illustrate them:

$$(\sqrt{a} + \sqrt{b})^2 = a + 2\sqrt{(ab)} + b.$$

$$(x^{\frac{1}{2}} - y^{\frac{1}{2}})^2 = x + y - 2\sqrt{xy}.$$

$$(6 + x)^2 = 36 + 12x + x^2.$$

$$(x - 1)^2 = x^2 - 2x + 1.$$

$$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2}.$$

$$\left(\frac{1}{x} - x\right)^2 = \frac{1}{x^2} + x^2 - 2.$$

$$(5 - \sqrt{2})^2 = 25 - 10\sqrt{2} + 2 = 27 - 14{\cdot}142\ldots$$
$$= 12{\cdot}858\ldots.$$

$$(1 - \sqrt{3})^2 = 4 - 2\sqrt{3} = 2(2 - \sqrt{3}) = {\cdot}536\ldots.$$

Notice that although $\sqrt{3}$ is greater than 1 the squared difference cannot help being positive.

$$\left(\frac{1}{\sqrt{3}} + \sqrt{3}\right)^2 = \tfrac{1}{3} + 2 + 3 = 5{\cdot}\dot{3}.$$

$$\left(\sqrt{5} - \frac{1}{\sqrt{5}}\right)^2 = 5 - 2 + \tfrac{1}{5} = 3{\cdot}2.$$

$$(121)^2 = (120)^2 + 240 + 1 = 14641.$$

$$(119)^2 = (120)^2 - 240 + 1 = 14161.$$

$$(1{\cdot}5)^2 = (1 + \tfrac{1}{2})^2 = 1 + 1 + \tfrac{1}{4} = 2{\cdot}25.$$

$$(1{\cdot}\dot{3})^2 = (1 + \tfrac{1}{3})^2 = 1 + \tfrac{2}{3} + \tfrac{1}{9}$$
$$= 1{\cdot}\dot{6} + {\cdot}\dot{1} = 1{\cdot}\dot{7}.$$

Problems.

1. If any diagram has all its linear dimensions increased by one-sixth, by how much is the whole area of the figure increased?

The answer liable to be given is one thirty-sixth, but it is not right. The right answer is $\frac{13}{36}$ths, or a little more than one-third of the original area. The first answer attends only to the little corner squares and neglects the two strips, for

$$(a+b)^2 - a^2 = b^2 + 2ab;$$

the $2ab$ being much bigger than b^2.

The simplest solution is to say that in the linear dimensions throughout, 6 has become 7, hence, in the area, 36 has become 49; wherefore the superficial increase is 13 of the same parts, that is 13/36ths of the original.

2. If a block is reduced in the ratio of 3 : 2 linear, that is if its length, breadth, and thickness are all made two-thirds of what they were, the shape being preserved, what change has been made in the surface or superficial area and in the volume or cubical contents?

Answer. The linear dimensions being reduced by one-third, or from 3 to 2, the superficial are reduced by five-ninths, or from 9 to 4; and the cubical are reduced by nineteen twenty-sevenths, or from 27 to 8. In other words the surface is less than half what it was, and the volume is less than a third what it was.

3. If every linear foot becomes 13 inches, every square foot becomes 169 square inches, and every cubic foot becomes 2197 cubic inches. So, while the linear increase is $\frac{1}{12}$th of the original, the superficial increase is $\frac{25}{144}$ths, or a little more than $\frac{1}{6}$th of the original area; the volume increase is $\frac{469}{1728}$ths, or distinctly more than $\frac{1}{4}$th of the original volume.

4. If one per cent. is docked off linear dimensions, about

two per cent. are thereby taken from area, and about three per cent. from bulk.

Now use the same equation to find square roots.

Suppose we want the square root of 50. We see instantly that it is a little more than 7, let us call it $7+x$, then write

$$50 = (7+x)^2 = 49+14x+x^2,$$

or, subtracting 49 from both sides (*i.e.* transferring 49 over to the left with change of sign),

$$1 = 14x+x^2,$$

wherefore $x = \frac{1}{14}$ is a first approximation, for the x^2 is a very small number, almost negligible. x is really a trifle less than $\frac{1}{14}$, though not so much less as $\frac{1}{15}$ would be, for its defect is $\frac{1}{14}$ of x^2, which is approximately $\frac{1}{2800}$ only. It is impossible to express the root accurately, and the result obtained by neglecting x^2 is usually a quite sufficiently close approximation. So the root is $7+\frac{1}{14} = 7{\cdot}0714$. The error is in the last place; the 4 is too big, it ought to be a 1.

So also to guess the root of 143

Write it $12-x$.

$$143 = (12-x)^2 = 144-24x+x^2,$$

neglect x^2, and $\qquad x = \frac{1}{24} = {\cdot}0417,$

so approximately $\qquad \sqrt{143} = 12-{\cdot}0417 = 11{\cdot}9583.$

Its real value is 11·9582607 ....

What is the square root of 99?

$$99 = (10-x)^2 = 100-20x+x^2,$$

so $\qquad x = \frac{1}{20} = {\cdot}05$

and $\qquad \sqrt{99} = 9{\cdot}95 \ldots.$

What is the root of 395?

Let it be $\qquad 20-x.$

$$395 = 400 - 40x+x^2$$

or $\qquad x = \frac{5}{40} = \frac{1}{8} = {\cdot}125,$

so $\qquad \sqrt{395} = 19{\cdot}875 \ldots.$

What is the square root of 1,000,015?

that is of 10^6+15.

Call it 10^3+x;

then $$10^6+15 = (10^3+x)^2 = 10^6+2000x+x^2\,;$$

whence $$x = \frac{15}{2000} = \cdot 0075,$$

and so the required root is 1000·0075.

In such a case the extra quantity is extremely small, and we see that in the root it is just half the value of the corresponding quantity in the given square.

This is a handy approximation which may be generalised and recollected. It is an immediate consequence of neglecting x^2 and writing $(1+x)^2 = 1+2x$ approximately when x is small.

So $$\sqrt{1+2x} = 1+x \quad \text{approximately.}$$

For instance $\sqrt{(1\cdot 008)}$ will equal 1·004;

and $$\sqrt{(100\cdot 084)} = 10\sqrt{1\cdot 00084} \triangleq 10\times 1\cdot 00042 = 10\cdot 0042\,;$$

or the equation may be written

$$\sqrt{(1+x)} = 1+\tfrac{1}{2}x \text{ approximately.}$$

The following relation,

$$\sqrt{(10^n+x)} \triangleq 10^{\frac{1}{2}n} + 10^{-\frac{1}{2}n}\,.\,\tfrac{1}{2}x,$$

when x is moderately small, is a general result; but for memory it is best to make the first term unity; and so in the numerical example just above, the factor 100 was first taken outside the root, where of course its value is 10. If the factor had been 1000 instead of 100, that is, if there had been an odd number of ciphers in it, this could not have been done so easily: we should then have had a $\sqrt{10}$ to deal with, and that would destroy the advantage of the process.

The process applies most obviously to numbers which can be separated into two very unequal portions, one of which has a known square root. If they are not very unequal, the neglect of x^2 becomes of more consequence, and the same sort of process must be continued further, before the square of an outstanding error is neglected.

Suppose for instance we wanted the square root of 72; we could write it as $(8+x)^2$ or we could write it as $(9-y)^2$,

so that $$72 = 64 + 16x + x^2,$$
or $$72 = 81 - 18y + y^2,$$
whence approximately $x = \frac{8}{16} = \frac{1}{2}$, and we should be neglecting $\frac{1}{4}$;
or $y = \frac{11}{18}$ and we should be neglecting a trifle less.

So the answer would be roughly 8·5,* but this would be a little too big, and the process must be continued, by successive approximations, beyond 8·4, in such a case; the process develops, in fact, into the ordinary arithmetical method of finding a square root, as described but not explained in the last chapter. We can now explain it, for it all depends on what we have just been doing; it involves an ultimate ignoring of an x^2, but it carries the process of surmising the root to any desired degree of approximation, before the inevitable outstanding error is considered so minute that its square may safely be neglected.

To illustrate the process arithmetically, and at the same time display its rationale algebraically, take any simple number at random, say for instance 33, call it N, and proceed to approximate to its square root.

(1) First guess the nearest lower integer root, namely 5, call it a in general, and write x for the unknown necessary complement to be found, so that

* In the particular example chosen it happens to be very easy to calculate the square root, because the 'factor method' would apply. Beginners may be reminded always to keep an eye open for the simple and satisfactory factor-method, such as this:

$$72 = 9 \times 8,$$
so $$\sqrt{72} = 3 \times \sqrt{8} = 3 \times 2\sqrt{2} = 6\sqrt{2} = 8{\cdot}48528 \ldots.$$

$$N = (a+x)^2 = a^2 + 2ax + x^2,$$

or

$$33 = (5+x)^2 = 25 + 10x + x^2.$$

From this we deduce that the deficiency $N - a^2 = x(2a+x)$,

or that $8 = x(10+x)$.

This gives us our first approximation to the required complement, or error in our rough estimate of the root, namely

$$x = \frac{8}{10+x}, \text{ or say } \cdot 7 \text{ as the first digit of it.}$$

(2) Thus we can now make a closer guess at the root, namely 5·7, and start afresh for a second approximation, x', writing

$$33 = (5\cdot 7 + x')^2 = 32\cdot 49 + 11\cdot 4x' + x'^2,$$

so the second deficiency is $\cdot 51 = x'(11\cdot 4 + x')$,

which gives, as the second outstanding error,

$$x' = \frac{\cdot 51}{11\cdot 4 + x'} = \cdot 04 \text{ as the first digit of that.}$$

(3) Our approximation to the root has now become 5·74, and we start off a third time to write

$$33 = (5\cdot 74 + x'')^2 = 32\cdot 9476 + 11\cdot 48x'' + x''^2,$$

whence the third deficiency $\cdot 0524 = x''(11\cdot 48 + x'')$, which gives us $x'' = \cdot 004$ as the next digit of the rapidly diminishing outstanding error.

(4) The approximation is now getting closer, being 5·744, and so we start again, saying

$$33 = (5\cdot 744 + x''')^2,$$

whence the fourth deficiency comes out

$$\cdot 006464 = x'''(11\cdot 488 + x'''),$$

yielding $x''' = \cdot 0005 \ldots$ as the error still remaining.

(5) We have now arrived at $\sqrt{33} = 5\cdot 7445 \ldots$, and we can continue the process as long as we like; but, at this (or at any other) stage, we can take refuge in simple division, to get at once a still closer approximation. For hitherto we have not neglected the square of *any* small quantity: everything so far

has been exact; but sooner or later exactness will have to be abandoned, because we know that a number really has no exact numerical root. It was considered too inaccurate to neglect the square of x, but we might perhaps have neglected the square of x', or at least of x''. We did not neglect even this however, but we are now going to neglect the square of x'''; so after reckoning the present deficiency, ·00071975, instead of saying

$$x'''' = \frac{\cdot 00071975}{11\cdot 4890 + x''''},$$

which would be continuing the process, we will say simply

$$x'''' = \frac{\cdot 00071975}{11\cdot 4890},$$

very nearly, and divide straight out, getting ·00006265 as the result. Wherefore finally the approximation at which we have arrived is $\sqrt{33} = 5\cdot 74456265 \ldots$.

If the process thus elaborated be compared with the operation as ordinarily performed, a little thought will make everything clear without more words.

The only thing that can require explanation is the actual mode of reckoning the successive outstanding deficiencies, viz.: $N - a^2$; $N - (a + x)^2$; $N - (a + x + x')^2$; and $N - (a + x + x' + x'')^2$. The original number N is not in practice thus manifestly reverted to for the purpose of getting these values—which in the above numerical example are successively

8; ·51; ·0524; and ·006464,—

but exactly the same result is obtained by the successive subtractions as ordinarily performed: the value of an expression like $(N - a^2) - x(2a + x)$ being practically employed, each time, instead of the equivalent $N - (a + x)^2$, because (having already found $N - a^2$) it is quicker to reckon.

The well known ordinary process is here exhibited for the same number, in order that it may be compared with the above fully explained treatment. To find the square root of 33, write

	33·	5·7445
	25·	
10·7	8·00	
	7·49	
11·44	·5100	
	·4576	
11·484	·052400	
	·045936	
11·4885	·00646400	
	·00574425	
11·4890	·00071975	

and the outstanding error in the root is very closely indeed equal to the residual deficiency divided by twice the root so far found, that is to say, ·00071975 ÷ 11·4890, or ·00006265.

The advantage of the approximation we noted on p. 295,

$$\sqrt{(1+y)} \bumpeq 1 + \tfrac{1}{2}y \ldots,$$

is so great that even when the first number is conspicuously not unity, it is often convenient to make it so by division. For instance to find $\sqrt{85}$, it equals $\sqrt{(81+4)}$

$$= 9\sqrt{(1+\tfrac{4}{81})} \bumpeq {}^{*}9(1+\tfrac{2}{81}) = 9 + \tfrac{2}{9} = 9{\cdot}\dot{2}.$$

And so with some of the other examples, they too may be done this way. We will therefore repeat them.

$$\sqrt{50} = \sqrt{(49+1)} = 7\sqrt{1+\tfrac{1}{49}} \bumpeq 7(1+\tfrac{1}{98})$$
$$= 7 \times 1{\cdot}0102 = 7{\cdot}0714.$$

In this case the approximate value ·0102 is obtained thus. 98 is two per cent. less than 100, so $\frac{1}{98}$ is two per cent. greater than ·01.

$$\sqrt{143} = \sqrt{(144-1)} = 12\sqrt{(1-\tfrac{1}{144})} \bumpeq 12(1-\tfrac{1}{288}) = 12 - \tfrac{1}{24}.$$

*At this stage the second term is halved and the root sign dropped.

$$\sqrt{99} = 10\sqrt{(1 - \tfrac{1}{100})} \simeq 10(1 - \tfrac{1}{200}) = 10 - \cdot 05 = 9\cdot 95.$$

$$\sqrt{375} = \sqrt{(400 - 5)} = 20\sqrt{(1 - \tfrac{1}{80})} \simeq 20(1 - \tfrac{1}{160})$$
$$= 20 - \tfrac{1}{8} = 20 - \cdot 125 = 19\cdot 875.$$

Perhaps decimals might be preferred throughout. Sometimes they would be handier, sometimes not.

$$\sqrt{396} = \sqrt{(400 - 4)} = 20\sqrt{(1 - \cdot 01)} \simeq 20(1 - \cdot 005)$$
$$= 20 - \cdot 1 = 19\cdot 900.$$

The result of this convenient approximation is always to give slightly too big a value for the root, and this whether terms under the root are separated by a negative or a positive sign.

Thus for instance the approximation to $\sqrt{101}$ namely 10·050, and to $\sqrt{99}$ namely 9·950 are both of them a trifle too big.

The error itself can be estimated by a further stage of approximation, and so gradually we can get as nearly accurate as we please, but we leave it there for the present.

The error in either case is about ·000125, so the digits as they stand above are fairly near the truth.

Cubes and Cube Root.

Now let us see what we can get of the same kind to help us in other cases. Suppose we cube a binomial, what shall we get?

First notice that

$$\begin{aligned}(a+b)(c+d)(e+f) &= (a+b)(ce+cf+de+df)\\ &= ace+acf+ade+adf\\ &\quad +bce+bcf+bde+bdf,\end{aligned}$$

eight terms altogether.

So take the three factors all alike.

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)(a+b)\\ &= aaa+aab+aba+abb\\ &\quad +baa+bab+bba+bbb\\ &= a^3+a^2b+a^2b+ab^2\\ &\quad +a^2b+ab^2+ab^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3.\end{aligned}$$

Not a very simple expression at first sight, but quite simple when you get accustomed to it, and very easy to remember and write down.

Notice first that every term is of the same "dimension," that is to say it involves three letters multiplied together, no more and no less. There is no term involving only a^2, nor only b^2, nor a alone, nor is there anything like a^4. The expression is a cube, and every term is of the nature of a cube. If a and b were lengths, the cube is a volume, and every term is necessarily a volume. You cannot with any sense add an area like a^2 to a volume like a^3, but you can add a volume like a^2b or like ab^2 to another volume like a^3, and you can add each more than once, in fact 3 times if you choose.

Notice next that the power of a decreases by one each term, and the power of b increases. We might, if we liked, introduce the index 0, because we know that

$$a^0 = 1 = b^0.$$

So the more fully written expression

$$a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$$

would represent, with needless explicitness, the truth that the sum of the indices of each term is 3.

As to the big 3's prefixed to the two middle terms, they are styled coefficients, or numerical factors; we have seen exactly how they arise, simply because we had to add three equal terms. They take the place of the 2 in the middle term when we were squaring a binomial.

We illustrated the square of a binomial by fig. 31,—where the a^2 and the b^2 and the two rectangles each equal to ab are obvious, and plainly make up the $(a+b)^2$.

So also we can geometrically illustrate the cube of a binomial; taking a cube whose every edge is divided into any two parts, respectively a and b. we get a figure like 33, which is more easily realised when built up or sawn out of wood.

Such of the portions as are visible are labelled with their respective volumes. There is first a big cube a^3, then there are three slabs each of area a^2 and thickness b, but one of them in the figure is invisible at the back; there are 3 rods or

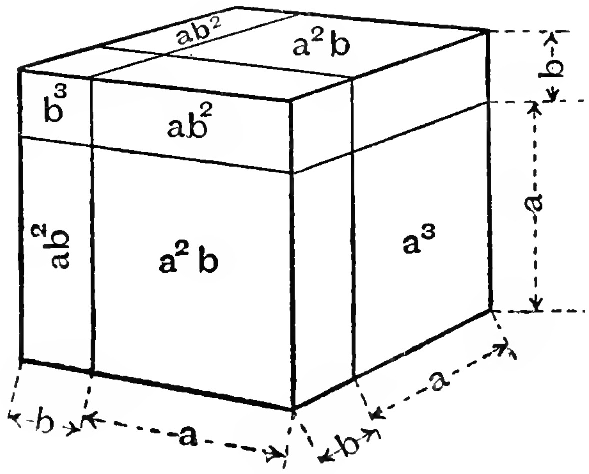


FIG. 33.

prisms each of the length a and sectional area b^2; and lastly there is a little cube b^3 diagonally opposite the big one; and these make up the 8 pieces, out of which the whole cube has been built up, $(a+b)^3$.

This then is a solid figure illustrating the cube of a binomial in the same sort of way that Euclid II. 4 illustrates the square of the same quantity.

Suppose we wished to illustrate the *fourth* power of a binomial by geometry. We could not possibly do it in any natural fashion, for we have already exhausted all the dimensions of space. Hence geometrical propositions on involution are not only complicated and wordy, but are feeble and limited.

Algebra is not limited at all; we can raise a binomial to the fourth, fifth, fifteenth, or any other power that we please, and presently we will do it. But first we will take a few examples and applications of what we have learnt about the cube or third power.

First a mere numerical illustration or verification:

$$\begin{aligned}(5+2)^3 &= 5^3 + (3\times 25\times 2) + (3\times 5\times 4) + 2^3\\ &= 125 + 150 + 60 + 8\\ &= 343.\end{aligned}$$

Then take a case where the first term is unity:

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3,$$

and then one with the second term negative:

$$(1-x)^3 = 1 - 3x + 3x^2 - x^3.$$

Notice in this case that the signs in the expansion are alternate, because the powers of $(-x)$ are alternately odd and even: the odd will all be negative, and the even will be positive. The general case, with the negative sign to the second member of the binomial, ought also to be recorded:

$$\begin{aligned}(a-b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3,\\ (x-1)^3 &= x^3 - 3x^2 + 3x - 1,\end{aligned}$$

but this is just the same as $(1-x)^3$ with the sign of every term reversed.

It is worth obtaining the general result for the third power of $a \pm b$ in another way, by help of what we know about its second power.

$$\begin{aligned}(a\pm b)^3 &= (a\pm b)(a^2 \pm 2ab + b^2)\\ &= a^3 \pm 3a^2b + 3ab^2 \pm b^3.\end{aligned}$$

Observe that the alternative sign affects only alternate terms, viz. those which involve the odd powers of the possibly negative quantity b. Among its even powers there is never any variety.

Another special case is

$$\left(x+\frac{1}{x}\right)^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}.$$

This is rather a curious case, considered from the point of view of the 'dimensions' of each term. x^3 looks like a volume, and would be a volume if x were a length, and

$3x$ would be merely treble that length; then come reciprocals. How can this be possible? Answer:—It is never possible to have different dimensions in different terms of an expression. It is quite easy and common to have *factors* of different dimensions, as components of a single term, united by the sign $\times$, but different terms united by the sign $+$ or $-$ are *always* of the same dimensions.

Apply that to the case of $\left(x+\frac{1}{x}\right)^3$ and we see that x^3 cannot possibly be a volume, nor can x be a length, if 1 is a pure number. It can in that case only be of the same dimension as its reciprocal.

Length and volume are all very well as illustrations, but it would be a great mistake to suppose that algebraic symbols can express nothing else. The terms "square" and "cube" suggest geometrical signification, and that doubtless was their original meaning, but now they have been so generalised that the original geometrical signification is almost forgotten. Cube is still used merely as short for "third power," and square is short for "second power," but the things that we raise to powers may be anything whatever that we find convenient. Often they are mere numbers, like a number of oranges. If we speak of 3^3 oranges meaning 27 oranges, it may be a pedantic mode of statement, but it is not incorrect. Even if we spoke of a cube of 3 oranges, or 3 oranges cubed, we might possibly be understood, as meaning a cubical box full of oranges with 3 in each edge, 9 in each face, and 27 in the box.

But this expression would not bear close examination, unless we put it in brackets, thus, (cube of 3) oranges, and then it does express more than merely 27. For 27 might be lying about anyhow, but (cube of 3) signifies that they are packed in a certain compact arrangement.

Why is cube of (3 oranges) wrong? Because that would mean $3^3 \times \text{oranges}^3$; and the latter factor has no meaning. Cube of (3 feet) is perfectly right, for that means

$$3^3 \times \text{feet}^3 = 27 \text{ cubic feet.}$$

You can have a cubic foot, but you cannot have a cubic orange; or rather perhaps you cannot have anything linear or superficial in oranges, as you can with feet or metres or inches.

Returning to the expression $x + \frac{1}{x}$ then, what can x mean? Only a thing whose dimensions are the same as its reciprocal, that is to say, a thing which has *no* "dimension," not a concrete thing at all, but an abstract number, a number of things abstracted from "things" altogether and contemplated alone. That is what we mean by an "abstract number" or "pure number." It is the simplest kind of "abstraction" there is, and the first we arrive at; later we shall employ plenty more.

If n is a pure number, like 2,

$\frac{1}{n}$ is likewise a pure number like $\frac{1}{2}$.

n^2 is also a pure number, and n^3, and any power.

$\sqrt{n}$ or any root is also a pure number.

So is $\log n$.

We cannot assert that a^n is a pure number for certain, because it depends entirely on what a is.

a might be a length, in which case a^2 would be an area, and a^3 be a volume. a^4 would in that case have no assignable physical meaning, but it would certainly not be a pure number.

There is therefore no difficulty about an expression like

$$\left(x - \frac{1}{x}\right)^3 = x^3 - 3x + \frac{3}{x} - \frac{1}{x^3} = \frac{(x^2-1)^3}{x^3},$$

every term must be a pure number.

This is not necessary with the next example, because there all the terms in the expansion have the same dimensions; provided always that a and b are quantities of similar kind.

$$(a^2 - b^2)^3 = a^6 - 3a^4b^2 + 3a^2b^4 - b^6.$$

In the next case, however, a must be a pure number, because of the term unity. If 1 means 1 something, the something cubed can go outside the brackets: it must apply equally to both a^2 and 1.

$$(a^2 - 1)^3 = a^6 - 3a^4 + 3a^2 - 1.$$

$$(1 - \surd 2)^3 = 1 - 3\surd 2 + (3 \times 2) - (\surd 2)^3$$
$$= 1 + 6 - 3\surd 2 - 2\surd 2$$
$$= 7 - 5\surd 2 = 7 - 7{\cdot}071 = -{\cdot}071.$$

$$(7{\cdot}1)^3 = 7^3 + {\cdot}3 \times 7^2 + 21 \times {\cdot}01 + {\cdot}001$$
$$= 343 + 3 \times 4{\cdot}9 + {\cdot}211$$
$$= 357{\cdot}911.$$

$$(57)^3 = (50 + 7)^3 = 125000 + (21 \times 2500) + (150 \times 49) + 343 = 185193;$$

but in this case it would be easier to do it by simple multiplication, $57 \times 57 \times 57$, or perhaps by logarithms. The worst of logarithms for finding a positive integer power is that they only give it approximately, unless you take a considerable number of places; and an integer power never is approximate, it can *always* be numerically expressed, because we start with a number and only multiply it by itself.

By "integer power" or "integral power" I do not mean a power of an integer, I mean any number raised to a power whose *index* is a whole number and not a fraction. If the index is fractional it represents a root. The case is entirely different with a root, for then we are endeavouring to find something which multiplied by itself will produce a given number: and the result is usually incommensurable.

But for integer indices, whether positive or negative, we can always get an exact result by straightforward multiplication; for instance 2^{24}, or 2^{-3}, or $(1{\cdot}2)^3$.

$$(1{\cdot}2)^3 = 1 + (3 \times {\cdot}2) \times (3 \times {\cdot}04) + {\cdot}008$$
$$= 1 + {\cdot}6 + {\cdot}12 + {\cdot}008$$
$$= 1{\cdot}728,$$

which is a familiar number—expressing the thousandth part of a cubic foot, if 1·2 means the tenth of a foot in inches.

Now find a *cube* root or two by the approximation method, choosing numbers which are not very different from a perfect cube.

Say we want the cube root of 65, call it $4 + x$.

$$65 = (4 + x)^3 = 64 + 48x + 12x^2 + x^3;$$

So the first approximation to x is $\frac{1}{48}$.

This however is a trifle too big, because $12x^2$ has been neglected. So we might call it $\frac{1}{49}$ or even $\frac{1}{50}$, at a shot, and say that the answer is 4·02 .... As to neglecting x^3 it is of slight consequence. This process, elaborated, is the basis of the arithmetical cube-root rule.

Take only one more example of finding cube roots, because they are usually done most easily by logarithms.

To find $\sqrt[3]{341}$.

$$341 = (7 - x)^3 = 343 - 3 \times 49x + 21x^2 - x^3;$$

$\therefore$ approximately $x = \dfrac{2}{3 \times 49}$, or, as this is a trifle too small,

say $\dfrac{2}{3 \times 48} = \dfrac{1}{72} = {\cdot}0139$. So approximately

$$\sqrt[3]{341} = 7 - {\cdot}0139 = 6{\cdot}9861,$$

which is still a trifle too small in the last place The digit 1 ought to be a 3 or a 4.

As an exercise it would be desirable to establish a method akin to the square root approximation, like this,

$$\begin{aligned}(341)^{\frac{1}{3}} &= (343-2)^{\frac{1}{3}} = 7(1-\tfrac{2}{343})^{\frac{1}{3}}\\ &= 7(1-\tfrac{2}{1029}) \text{ approximately}\\ &= 7\times\tfrac{1027}{1029}\text{; which equals ·2 per cent. less than 7,}\\ &= 6{\cdot}986 \text{ roughly;}\end{aligned}$$

or generally, when x is a small quantity,

$$\sqrt[3]{(1\pm x)} = 1\pm\tfrac{1}{3}x \text{ approximately;}$$

which is equivalent to neglecting squares and cubes and all higher powers of x.

Approximations.

The fact that the square of a small quantity is very small, and the cube of it extremely small, is easy enough to understand; and since it is extremely useful in application, it should be thoroughly understood and remembered. Let the small quantity be 1 per cent., for instance, or ·01 or $\frac{1}{100}$. Its square is $\frac{1}{10000}$, one ten-thousandth; and its cube is a millionth;

If then we have to find $(1{\cdot}01)^3$, it will

$$\begin{aligned}&= 1 + {\cdot}03 + {\cdot}0009 + {\cdot}000001\\ &= 1{\cdot}030901,\end{aligned}$$

of which the first significant digit of the decimal is decidedly the most important, the second is sometimes worth attention, denoting a value about $\frac{1}{33}$rd of the previous one, and the last is utterly trivial, except for exact mathematical purposes.

A cube of a foot and one inch (or 13 inches cubed), (13 inches)^3, is decidedly bigger than a cubic foot; but nevertheless a cubic inch is almost negligible in comparison with a cubic foot: it is only the $\frac{1}{1728}$th part of it.

Let us examine this, because beginners often make mistakes here.

(1 foot + 1 inch)^3 they incline to write down as a cubic foot plus a cubic inch: which is just the mistake of thinking that $(a+b)^3$ equals a^3+b^3; in other words it is the mistake of altogether ignoring $3a^2b+3ab^2$, three slabs and three rods, and attending only to the little insignificant corner cube of the small quantity b (supposing b to be a small quantity) in fig. 33.

The true value is

$$\begin{aligned}(1 \text{ foot} + 1 \text{ inch})^3 &= (1 \text{ foot})^3 + 3(\text{feet})^2 \times 1 \text{ inch} \\ &\qquad + 3 \text{ feet} \times (\text{inch})^2 + (1 \text{ inch})^3 \\ &= 1 \text{ cubic foot} \\ &\quad + 3 \text{ slabs a foot square and an inch thick} \\ &\quad + 3 \text{ rods a foot long and a square inch section} \\ &\quad + \text{a cubic inch.}\end{aligned}$$

The last term is the most trivial of the eight terms, and the 3 slabs are the most important after the cubic foot itself.

Translating to inches, we see that

$$\begin{aligned}(13 \text{ inches})^3 &= 1728 + (3 \times 144) + (3 \times 12) + 1 \\ &= 2197 \text{ cubic inches,}\end{aligned}$$

which is otherwise very easily arrived at.

If instead of a foot and an inch we had taken a yard and an inch, the smallness of everything except the slabs would have been accentuated; and if we take a metre and a millimetre we shall see it still more forcibly:

$$\begin{aligned}(1 \text{ metre} + 1 \text{ millimetre})^3 = {} & 1 \text{ cubic metre} + 3 \text{ slabs a metre} \\ & \quad \text{square and a millimetre thick} \\ & + 3 \text{ lines a metre long and a} \\ & \quad \text{square millimetre cross section} \\ & + \text{a millimetre cube;}\end{aligned}$$

or expressing it all in cubic centimetres

$$\begin{aligned}&= 1 \text{ million c.c.} + \text{three hundred thousand c.c.} + \text{three c.c.} \\ &\qquad + \text{a thousandth of a c.c.} \\ &= 1{,}300{,}003{\cdot}001 \text{ c.c.}\end{aligned}$$

When things expand by heat, the expansion is usually very small; the increase of bulk is not so small as the increase of length however. If the edge of a cube expands 1 per cent. the volume of it expands just about 3 per cent., and the area of one of its faces about 2 per cent. This follows from what we have been saying. Compare page 293, No. 4.

It is sometimes expressed by saying that the proportional superficial expansion is twice the linear, while the cubical expansion is three times the linear. We will employ the subject of expansion to furnish us with a few interesting arithmetical examples of an easy and uncommercial kind in a future chapter, but first we will do some algebraic expansions.

CHAPTER XXXIV.

To find any power of a Binomial.

SUPPOSE we have to find $(a+b)^4$, we have only to multiply $a+b$ by itself four times, and write down the result. We might write it thus

$$\begin{aligned} &(a+b)^2(a+b)^2 \\ =\ &(a^2+2ab+b^2)(a^2+2ab+b^2) \\ =\ &a^4+2a^3b+\ a^2b^2 \\ &\quad +2a^3b+4a^2b^2+2ab^3 \\ &\quad\quad +\ a^2b^2+2ab^3+b^4 \\ \hline =\ &a^4+4a^3b+6a^2b^2+4ab^3+b^4 \end{aligned}$$

Now here we see the same sort of law as was observed in the expansion of $(a+b)^3$; the indices of a decrease regularly, and those of b increase regularly, so that every term is of the fourth degree. The numerical coefficients follow a less obvious law. Let us write them down for the cases that we know.

for $(a+b)$				1		1				
,, $(a+b)^2$			1		2		1			
,, $(a+b)^3$		1		3		3		1		
,, $(a+b)^4$	1		4		6		4		1	

The law is fairly plain, and we might guess the coefficients for the next sets :

$(a+b)^5$		1	5	10	10	5	1
$(a+b)^6$	1	6	15	20	15	6	1

and then we can verify them, by direct multiplication, thus

$$(a+b)^5 = (a+b)^3(a+b)^2$$
$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5,$$
$$(a+b)^6 = (a+b)^3(a+b)^3$$
$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

A guessing process like the above, which is subsequently verified and obviously extensible to the case of any positive integer as index, is a method of frequent and considerable use in order to first ascertain a rule or law or method of procedure; but one should not rest satisfied without perceiving the rationale of it, and so to say "proving" it or reasoning it out; otherwise it remains what is called an "empirical" law, meaning a law ascertained by experiment and observation without a full knowledge of the reason. Some laws have to remain of this character, when the subject matter is difficult or obscure; but that is not the case with little calculations like the present: the reasonableness of the result can always be made out, and it is a most wholesome exercise. In the present instance the method of expanding any binomial as an empirical process seems to have occurred to Isaac Newton while still quite young; and the reasoned proof of this process is what we now know as "the binomial theorem."

We will not go into this fully just at present, nor at all more fully than is needed for practical purposes, but for a positive integer the empirical process itself is easy and worth while for anybody to know.

First write down what we have observed, for any positive integer index n, concerning $(a+b)^n$:—

We know that the powers of a will begin with a^n, and decrease by one each time down to a^0 or unity.

The powers of b will begin with b^0, or unity, and climb by one each time up to b^n: so that as regards the algebraic part of the expansion, the terms will be

$$a^n,\ a^{n-1}b,\ a^{n-2}b^2,\ a^{n-3}b^3,\ \ldots\ldots\ a^2b^{n-2},\ ab^{n-1},\ b^n,$$

the sum of the indices of a and b always adding up to n, which may be called the "order" or "degree" of the whole.

Now what about the numerical coefficients? We can obtain them as follows. Take the coefficient of any term, multiply it by the index of a in that term, and divide by the number of terms preceding the next term, the result will give the coefficient for that next term. This is what we have ascertained empirically, though we did not word or express it before, but it is what we did or might have done; because, take the case of $(a+b)^5$,

the first term is a^5,

so the coefficient of the next term is 5, giving

$$a^5 + 5a^4b.$$

Now take the 5 and the 4, multiply them together, and divide by 2; we get 10, which is the coefficient of the next term, carrying us as far as three terms,

$$a^5 + 5a^4b + 10a^3b^2.$$

Then take the 10 and the 3, multiply them, and divide by the number of terms; thus we get the next coefficient, viz. 10,

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3.$$

Now take the 10 and the 2, multiply them, and divide by 4, and we get the coefficient of the next term, viz. 5.

Then take 5 and 1, multiply, and divide by 5, and we get the coefficient of the last term, viz. 1, giving the whole expansion, with six terms in all,

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

In the last term the index of a is zero, hence a does not appear, because $a^0 = 1$; and if we apply the rule further it will give us zero as a factor of the next and of every succeeding term; which therefore all vanish, so the series terminates.

Try this rule also for $(a+b)^6$ and $(a+b)^7$, getting the result in the latter case,

$$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7,$$

and then apply it to $(a+b)^n$,

$$a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2 \,.\, 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{2 \,.\, 3 \,.\, 4}a^{n-4}b^4 + \text{etc.}$$

Now this is a most interesting example of a very important algebraic thing called a 'series.' It appears to go on for ever, but, as we have seen, it does not go on very long when n is a positive integer, for sooner or later there will come the index $n-n$, whose value is 0; and as this quantity $n-n$ will enter as a factor into every subsequent coefficient, they all vanish, and the term with index 0 applied to a is the last term.

Thus for $(a+b)^5$ there were six terms in the series, and no more. For $(a+b)^6$ there were seven terms, and no more; and for $(a+b)^n$ there will be $n+1$ terms, and no more, provided n is a positive integer. All subsequent terms are zero, because they all contain the factor $n-n$. But if the index n is a fraction, or if it has a negative value, even a negative integer value, the cause of stoppage will no longer occur; for, naturally, a *number-of-terms* can never be a fraction or negative. There will therefore never be an index $n-n$; there will be $n-7$, $n-8$, $n-9$, etc., but none of these can possibly be zero unless n itself is a positive integer.

Consequently in these cases the series does not stop, but goes on for ever, extending to infinity. It may happen however that its later terms become insignificantly small, and that all after a certain number can be neglected for practical purposes. This a point to which attention must be specially directed, because it is exceedingly useful in practice.

Notice that we have not yet established or *proved* the above series for the expansion or power of a binomial, even for the case of n a positive integer. We will defer the proof for the present: so far we have only arrived at it by experiment. The proof is not difficult for n a positive integer, but it will come better later. Mathematicians know how to prove it for a fractional and a negative index, that is for the case of an infinite series, which however is exactly of the same algebraic form as the one we have written.

The method of experiment and observation is quite a good practical method, only it might in some cases lead us wrong unless it can be checked over and reasoned out by some more intellectual process.

For the present we will accept the series and study it. Notice first the *denominators* of the several terms. They consist of a series of consecutive natural numbers 1 . 2 . 3 . 4 . 5, etc., multiplied together. This sort of product often occurs, and it is convenient to have a symbol for it. $\underline{|5}$ is the way it is written, 5 ! is the way it is printed, and it is called "factorial 5." They all mean the same thing, viz. $1 \times 2 \times 3 \times 4 \times 5$, that is to say 120.

So 4 ! has the value 24, since it means $1 \times 2 \times 3 \times 4$

$\underline{|3}$	,,	,,	6,
$\underline{|2}$	,,	,,	2,
$\underline{|1}$	,,	,,	1,
$\underline{|6}$	,,	,,	720,
7 ! or $\underline{|7}$	equals		5040,
8 !	=		40,320,

and so on. So that factorial 20 is an enormous number.

Now look at the numerators. They too are factorials of a kind, but they do not begin at 1, they begin at the other end; they represent a part of factorial n, with the early part cut off.

They might be denoted by $\dfrac{\lfloor n}{\lfloor n-r}$; for

$$n \,.\, n-1 \,.\, n-2 \,.\, n-3 \ldots n-r+1 = \frac{n!}{(n-r)!}$$

So the successive numerators are as follows:

$$\frac{\lfloor n}{\lfloor n}, \quad \frac{\lfloor n}{\lfloor n-1}, \quad \frac{\lfloor n}{\lfloor n-2}, \quad \frac{\lfloor n}{\lfloor n-3}, \text{ etc.,}$$

being 1, n, $n(n-1)$, $n(n-1)(n-2)$, etc., respectively.

Hence any of the coefficients may be written in this form

$$\frac{\lfloor n}{\lfloor n-r \, \lfloor r};$$

while as to the *ab* part corresponding to this general coefficient it will be $a^{n-r}b^r$.

Hence the whole series may be neatly written as the sum of a number of terms all of this kind, for every value of r from 0 to n; and such a summation is usually expressed by the capital letter sigma; hence

$$(a+b)^n = \sum_{r=0}^{r=n} \left\{ \frac{n!}{(n-r)!\, r!} a^{n-r} b^r \right\},$$

which means that you write down all the terms of this form in regular order from $r = 0$ up to $r = n$, and then add them together. Try to do this, for different values of n, for instance 3 or 4 or 5 or 6, and see that you get the series already obtained.

The only thing that requires explanation, until we come to fractional and negative indices, is how to interpret "factorial nought." To common sense such an expression sounds meaningless; and to understand it fully, together with the factorials of negative and fractional numbers, a good deal of mathematics must be conquered. It is easy however to show that $\lfloor 0$ must be interpreted as unity, that is to say that $\lfloor 1$ and $\lfloor 0$ are alike equal to 1.

Proof. $\underline{|n} = n\underline{|n-1}$; but in the special case when $n = 1$ n and $\underline{|n}$ are the same thing; hence in that case $\underline{|n-1}$ is unity, but it is also factorial nought.

Exercise.—Make a table of binomial coefficients up to say the index 12 as the finish. For answer, see p. 334.

A special case of frequent occurrence is when one of the terms of the binomial is unity, as for instance $(1+x)^n$.

Consider this case. Any binomial can be thrown into this form by an obvious process, as follows:

$$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n = a^n(1+x)^n,$$

where a^n is a factor taken outside brackets, and the ratio b/a is treated as a single quantity, a pure number, and called x.

Observe that a and b might be anything, so long as they are the same thing, but that x must be a number, in order that it may be added to 1; and being a ratio of similar quantities, it *is* a number. The most important case, with fractional and negative indices, is when x is a *small* number, for then the series, or expansion in powers of x, will rapidly diminish, and all beyond a few terms can be neglected. The meaning of this will become clearer soon.

First apply the ordinary rule for the expansion, observing that 1^{n-1}, 1^{n-2}, etc., need not be written, because they are all mere unity factors. We have nothing therefore to write but the successive binomial coefficients and the ascending powers of x.

$$(1+x)^n = 1+nx+\frac{n \,.\, n-1}{2!}x^2+\frac{n \,.\, n-1 \,.\, n-2}{\underline{|3}}x^3+\ \dots,$$

a very useful expansion; and if x is really small, so that x^2 may be neglected, it gives us this extremely handy approximation,

$$(1+x)^n \simeq 1+nx \quad \text{when } x \text{ is very small.}$$

As a matter of fact we have used this already for extracting approximate roots (p. 308), arriving at it by a different process.

Thus to find $\sqrt{(1+x)}$ when x is small we have only to put $n = \frac{1}{2}$.

$$\sqrt{(1+x)} = (1+x)^{\frac{1}{2}} \doteq 1 + \tfrac{1}{2}x \text{ approximately.}$$

E.g. $\sqrt{(1\cdot01)} = 1\cdot005$ approximately.

$$\sqrt{(1\cdot008)} = 1\cdot004 \quad \text{,,}$$

$$\sqrt{100\cdot6} = 10\sqrt{(1+\cdot006)} \doteq 10 \times 1\cdot003 = 10\cdot03.$$

So also for cube or other roots.

$$\sqrt[3]{(1+x)} = (1+x)^{\frac{1}{3}} \doteq 1 + \tfrac{1}{3}x,$$

$$\sqrt[3]{1003} = 10\sqrt[3]{1\cdot003} = 10(1\cdot003)^{\frac{1}{3}} \doteq 10 \times 1\cdot001 = 10\cdot01.$$

$$\sqrt[5]{33} = 2\sqrt[5]{(1+\tfrac{1}{32})} = 2(1+\tfrac{1}{32})^{\frac{1}{5}} \doteq 2(1+\tfrac{1}{160})$$
$$= 2 + \tfrac{1}{80} = 2\cdot0125.$$

Or take an example of a negative index.

$$\frac{1}{\sqrt{(1+x)}} = (1+x)^{-\frac{1}{2}} \doteq 1 - \tfrac{1}{2}x.$$

But this case of a negative index will bear examining more fully.

Let us write $n = -m$, and then interpret the general expression for the special case of a negative index. Observe that it is no new expansion, only the old one re-written with the sign of the index changed, but it looks different:

$$(a+b)^{-m} = a^{-m} + (-m)a^{-m-1}b + \frac{-m(-m-1)}{1\,.\,2}a^{-m-2}b^2 + \ldots$$

$$= a^{-m} - ma^{-(m+1)}b + \frac{m\,.\,m+1}{2!}a^{-(m+2)}b^2$$
$$- \frac{m(1+m)(2+m)}{\underline{|3}}a^{-(m+3)}b^3$$

$$= \frac{1}{a^m} - \frac{mb}{a^{m+1}} + \frac{m\,.\,m+1\,.\,b^2}{\underline{|2}\;a^{m+2}} - \frac{m(m+1)(m+2)}{\underline{|3}\;a^{m+3}}b^3 + \ldots \text{ etc.,}$$

the terms having alternate signs.

$(a+b)^{-m}$ would be similar but have all the terms positive.

Hence also

$$(1 \pm x)^{-m} = 1 \mp mx + \frac{m \,.\, m+1}{\underline{|2}} x^2 \mp \frac{m \,.\, m+1 \,.\, m+2}{\underline{|3}} x^3 + \mp \text{ etc.},$$

where it will be observed that with the + sign on the left, the terms on the right are alternately + and − ; but with the − sign on the left they are all + on the right.

The series is infinite, but if x is small a few terms practically suffice.

Examples.

Take some examples or special cases :

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + \frac{1 \,.\, 2}{\underline{|2}} x^2 - \frac{1 \,.\, 2 \,.\, 3}{\underline{|3}} x^3 + \ldots$$

$$= 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots,$$

of which only a few terms are important if x is small, *e.g.*

$$\frac{1}{1{\cdot}01} = (1{\cdot}01)^{-1} = 1 - {\cdot}01 + {\cdot}0001 - {\cdot}000001 + \ldots$$

$$= {\cdot}990099 \ldots \triangleq {\cdot}9901,$$

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots,$$

$$\sqrt{(1+x)} = (1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{\underline{|2}} x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{\underline{|3}} x^3 + \ldots$$

$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \ldots$$

$$\sqrt{1-x} = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 - \ldots,$$

$$\frac{1}{\sqrt{(1+x)}} = (1+x)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}x + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{\underline{|2}} x^2 + \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{\underline{|3}} x^3$$

$$= 1 - \tfrac{1}{2}x + \tfrac{3}{8}x^2 - \tfrac{5}{16}x^2 + \ldots,$$

$$\frac{1}{\sqrt{(1-x)}} = 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \tfrac{5}{16}x^3 + \ldots.$$

A curious case is afforded when both terms of the binomial are unity, like $(1+1)^n$. When the index n is not a positive integer the series is divergent and useless; but when n is a positive integer it is simple enough, for the sum is finite. It is a mere curiosity, but we may as well find a power in this way.

For instance to find 2^5,

$$(1+1)^5 = 1+5+\frac{5 \cdot 4}{\lfloor 2} + \frac{5 \cdot 4 \cdot 3}{\lfloor 3} + \frac{5 \cdot 4 \cdot 3 \cdot 2}{\lfloor 4} + \frac{\lfloor 5}{\lfloor 5}$$

$$= 1+5+10+10+5+1 = 32.$$

Similarly $2^6 = 1+6+15+20+15+6+1 = 64.$

This set of numbers, as tabulated in their early stages on pages 311 and 334, are called the binomial coefficients; and you observe that each set of them adds up to a power of 2. We had not noticed this before.

Now what is the good of an expansion generally? Is it of any practical use? Well it is, but it is the first few terms which are the most useful. The expansion of some power of $(1+x)$ is specially useful when x is very small, for then

$$\sqrt{(1+x)} \simeq 1+\tfrac{1}{2}x,$$

$$\sqrt{(1-x)} \simeq 1-\tfrac{1}{2}x,$$

$$\frac{1}{\sqrt{(1+x)}} \simeq 1-\tfrac{1}{2}x,$$

$$\frac{1}{\sqrt{(1-x)}} \simeq 1+\tfrac{1}{2}x.$$

This approximation is said to be correct to the first order of small quantities, or to be an approximation of the first order.

To be correct to the second order of small quantities we must introduce the terms involving x^2, and so on.

When x is only moderately small, third and even fourth terms may have to be employed, and the more terms introduced the more accurate will the result be.

If x is greater than 1, the series becomes hopeless, but if x is only slightly less than 1, it can always be approximated to sufficiently, by taking enough terms, though it is not then really useful.

The series is said to be convergent or converging when x is less than 1. **A converging series is one whose terms continually decrease in such a way that the sum of an infinite number of them is finite.**

For instance, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$ is a converging series, and its value, to an infinite number of terms, is 2;

but $$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \dots$$

happens not to be convergent, for though the terms keep on diminishing, they do not diminish with sufficient rapidity to be able to stop at any point and say 'we will neglect the rest.' Those which we neglected would in fact amount to more than those we took into account, for the sum of an infinite number of terms of such a series is infinite. It is not a convergent series at all, although each term is smaller than the preceding one. A curious case.

The first is called a geometrical progression, the second is called a harmonic progression, because it gives the series of the harmonics or simplest overtones in music. The time of vibration of the fundamental note being called 1, a trained ear can hear, when a string is struck or plucked or bowed, or when an open organ pipe is blown, other superposed notes, with their times of vibration $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., of the first; and these superposed or secondary tones are called harmonics. So the series is called a harmonic series.

An arithmetical series is one whose terms proceed by simple addition. In a harmonic series it is the denominators or reciprocals of the terms which proceed in this way. Fortunately we seldom or never want to *sum* a harmonic series.

CHAPTER XXXV.

Progressions.

WE have now, in the last chapter, arrived at an example of a series or progression. The subject of 'series' is immense and endless, but there are a few simple ones which are exceptionally easy to deal with.

Of these, three are commonly treated quite early, viz. the three called Arithmetical, Geometric, Harmonic, respectively.

In an arithmetical series the terms proceed by a common difference.

In a geometric series the terms proceed by a common factor.

In a harmonic series the reciprocals proceed by a common difference.

Thus 1, 2, 3, 4, 5, ... is the simplest example of an A.P.

1, 2, 4, 8, 16, ... „ „ „ G.P.

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ „ „ „ H.P.

But the common difference may be negative, or the common factor less than 1, so that

$7, 6, 5, 4, 3, 2, 1, 0, -1, -2, \ldots$ is an example of A.P.

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ „ „ G.P.

$\frac{1}{7}, \frac{1}{5}, \frac{1}{3}, 1, -\frac{1}{3}, -\frac{1}{5}, \ldots$ „ „ H.P.

Also 1, 1·25, 1·5, 1·75, 2 ... is an A.P.

1000, 100, 10, 1, ·1, ·01, ... is a G.P.

$\frac{1}{216} + \frac{1}{36} + \frac{1}{6} + 1 + 6 + 36 + 216 \ldots$ is also a G.P.

$-1, \; -1\cdot6, \; -5, \; 5, \; 1\cdot\dot{6}, \; 1, \; \cdot\dot{7}1428\dot{5}, \; \cdot\dot{5}, \; \cdot\dot{4}\dot{5}$ is an H.P.

The latter is perhaps too much disguised for a beginner, but if the terms be written as vulgar fractions it is plain enough: the denominators are in A.P. with common difference 2, for it is the same as

$$\frac{5}{-5}, \frac{5}{-3}, \frac{5}{-1}, \frac{5}{1}, \frac{5}{3}, \frac{5}{5}, \frac{5}{7}, \frac{5}{9}, \frac{5}{11}.$$

The thing that generally has to be done with a series is to evaluate the sum of its terms; and the most important are those whose terms decrease, so that an infinite number of them have a finite sum, which can be ascertained. Otherwise we must know how many terms we are intended to add up.

Another thing that may be necessary to do with a series, especially those which do not converge, or which actually increase as they go on, is to find the value of the nth term.

Thus in the horse-shoe nail question, page 155, we had really to add 24 terms of a G.P., beginning with 1 and proceeding by a common factor 2; but the finding of the 25th term of that series was sufficient, because we could see experimentally that each term was almost precisely equal to the sum of all that had preceded it, being, as a matter of fact, just *one* in excess. (Compare page 218.)

$1+2$	with 1 added	made	the next	term	4.
$1+2+4$	,,	,,	,,	,,	8.
$1+2+4+8$	,,	,,	,,	,,	16.
$1+2+4+8+16$	,,	,,	,,	,,	32.
etc.					

So all the first 24 terms, with 1 added, were equal to the 25th term of the series.

But the 25th term was 2^{24}, therefore the whole 24 terms of the series, added up, was one less than 2^{24}; that is to say the series equalled $2^{24} - 1$.

Of the three progressions we have mentioned, G.P. is certainly the most commonly occurring and the most useful. Let us take it first.

Let the common factor be called r, and the first term a, so that the terms run thus

$$a + ar + ar^2 + ar^3 + \dots,$$

r being any number whole or fractional.

If r, when interpreted arithmetically, is a negative quantity the terms will have alternately opposite signs, and the result will be a combination of alternate addition and subtraction; which however can conveniently be called the *algebraic* sum, meaning the sum when written algebraically with sign implied but unexpressed, but of course subtracting from the series those terms with negative signs when arithmetical interpretation is entered upon.

One sees at once that since the second term is

	ar
and the third	ar^2,
the fourth	ar^3,

the nth term must be ar^{n-1}.

The sum of the first n terms will therefore be

$$a(1 + r + r^2 + \dots + r^{n-1}).$$

Now this is a thing we have already come across; it was $\frac{1}{1-r}$ when r was small, that is to say

$$(1-r)^{-1} = 1 + r + \frac{(-1)(-2)}{\underline{|2}} r^2 - \frac{(-1)(-2)(-3)}{\underline{|3}} r^3 \dots$$

$$= 1 + r + r^2 + r^3 \dots,$$

but r must be less than 1, or the series will not converge; every term will get bigger than the preceding one if r is greater than 1, and there would be no meaning except infinity in an infinite number of such terms. But the expansion is only true for an infinite number of terms; consequently it is only serviceable when r is less than 1.

However, that is a very important case: the most important case. Let us apply that to a few examples before we go further.

Find the sum $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\ldots.$

Here $a = 1$ and $r = \frac{1}{2}$,

the sum then will be $1 \times \left(\frac{1}{1-r}\right) = \frac{1}{1-\frac{1}{2}} = 2$, which we already knew. (pp. 321 and 100.)

This series can be well illustrated by cutting up an apple or a loaf of bread; for if such an object be taken and first a half cut off, then a quarter, then an eighth, then a sixteenth, and so on, all the cutting can be performed on a single object, and however long the cutting be continued the single unit will not be exhausted: and yet if the cutting be continued *ad infinitum* the apple will be all exactly used up. In other words, although the sum of any finite number of terms of the series $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., is less than unity, the sum of the infinite series of these fractions is exactly one whole, no more and no less, that is to say

$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\ldots \textit{ ad inf.} = 1.$$

As another example take

$$1 + \cdot 1 + \cdot 01 + \cdot 001 + \ldots;$$

it equals $\frac{1}{1-\cdot 1} = \frac{1}{\cdot 9} = \frac{10}{9} = 1\cdot\dot{1}$,

as is otherwise obvious by simple addition of the terms.

Again $12+4+\frac{4}{3}+\frac{4}{9}+\frac{4}{27}+\frac{4}{81}+\ldots.$

The sum equals $\frac{12}{1-\frac{1}{3}} = \frac{12}{\frac{2}{3}} = 12 \times \frac{3}{2} = 18.$

Another way of putting it is to say that

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}.$$

The series $10 + 9 + 8{\cdot}1 + 7{\cdot}29 + \ldots$, to infinity, is a G.P. that does not decrease very fast, but it converges nevertheless, and the value towards which it converges, constantly approaching though never actually reaching, is $\frac{a}{1-r}$ as usual, that is 100;

$$\text{the sum to infinity} = \frac{10}{1-\frac{9}{10}} = \frac{100}{10-9} = 100 \text{ exactly.}$$

In general, so long as r is less than 1, it matters not how little less, the series will converge, and we can find the sum of an infinite number of terms. Suppose the common ratio were ·999 for instance, and the first term were 1, the sum to infinity would be $\frac{1}{1-\frac{999}{1000}}$, that is to say 1000. If the first term of this series had been anything else than 1, say 56 for instance, the sum would have been merely 56000. Or if the first term had been 4·35782, or any number you please, the sum would have been 4357·82, if the common factor were ·999 as supposed.

The first term therefore causes no difficulty, it is the common ratio or factor that requires attention when a finite number of terms is wanted; and a finite number of terms always is required whenever the common factor is greater than 1, and often is when it is less.

How are we to find the sum of n terms then?

It can be done by a contrivance:

Write down the series, and then write it down again with every term multiplied by r, and then subtract the two series, thus:

Call S the sum of n consecutive terms of the series.

$$S = 1 + r + r^2 + r^3 + \ldots r^{n-1};$$

$$\therefore\ rS = \quad r + r^2 + r^3 + \ldots r^{n-1} + r^n.$$

Now subtract

$$S - rS = 1 - r^n, \text{ because all the other terms go out;}$$

therefore $$S = \frac{1 - r^n}{1 - r}, \text{ or } \frac{r^n - 1}{r - 1},$$
which is the same thing.

If the first term is a, then the above expression has to be multiplied by a; so that in general, whatever r may be, the sum of n terms of a geometrical progression is

$$S = a\frac{r^n - 1}{r - 1}.$$

If n should be ∞ there is no finite meaning in the series unless r is less than 1; in that case $r^n = 0$, because higher powers of a proper fraction keep on diminishing, so an infinite power must disappear altogether; we then get the case which we already know, viz. $\frac{a}{1-r}$.

Examples.

Apply this to the sum of 24 horse-shoe nails with one farthing for the first, and with common factor 2. (p. 156.)

Ans.: The price is $a\frac{2^{24} - 1}{2 - 1} = 2^{24} - 1$ farthings.

Find the sum of six terms of the series

$$100 + 200 + 400 + \text{etc.}$$

Ans.: It equals $100 \times \frac{2^6 - 1}{2 - 1} = 6300.$

Find the sum of $1 + 3 + 9 + 27 +$ etc. to six terms.

Ans.: The sum $= \frac{3^6 - 1}{3 - 1} = \frac{728}{2} = 364.$

Find the value of $64 + 16 + 4 + 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64}$.

Ans.: This is a G.P. of seven terms with common ratio $\frac{1}{4}$ and first term 64.

So $$S = 64\frac{1 - (\frac{1}{4})^7}{1 - \frac{1}{4}} = \frac{4}{3} \times 64 \times \left(1 - \frac{1}{16384}\right)$$
$$\simeq \frac{256}{3}.$$

The numerator of the fraction $\frac{1-r^n}{1-r}$, in a case of many terms with a fractional ratio, is of small significance: it is nearly unity.

Algebraic Digression.

The result we have arrived at, as the sum of a G.P., may be regarded as an expansion for an algebraical division.

$$\frac{1-r^n}{1-r} = 1 + r + r^2 + r^3 + \ldots .$$

This might be generalised hypothetically thus,

$$\frac{x^n - y^n}{x-y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots ;$$

which could be verified by direct division, or more easily by multiplication, and could be led up to experimentally thus:

$$\frac{x^2-y^2}{x-y} = x + y,$$

$$\frac{x^3-y^3}{x-y} = x^2 + xy + y^2,$$

$$\frac{x^4-y^4}{x-y} = x^3 + x^2y + xy^2 + y^3,$$

and so on.

If we try the positive sign between the terms on the left, the matter is a little more troublesome.

Try it first in the denominator only:

$$\frac{x^2-y^2}{x+y} = x - y,$$

$\frac{x^3-y^3}{x+y}$ will not go without a remainder,

$$\frac{x^4-y^4}{x+y} = x^3 - x^2y + xy^2 - y^3,$$

$\frac{x^5-y^5}{x+y}$ will not go again,

and so on.

Now try the positive sign in numerator only:

$\dfrac{x^2+y^2}{x-y}$ will not go, *i.e.* will have a remainder;

$\dfrac{x^3+y^3}{x-y}$ will not go either.

Now try positive signs in both numerator and denominator:

$\dfrac{x^2+y^2}{x+y}$ will not go,

$$\frac{x^3+y^3}{x+y}=x^2-xy+y^2;$$

$\dfrac{x^4+y^4}{x+y}$ will not go,

$$\frac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4.$$

So it makes all the difference whether the indices are even or odd. All the above can easily be verified by direct operation; and the reason of the failure to divide out, when they do fail, will also be manifest on trial. The reason is that the last term, the y^3 or y^4, etc., would have the wrong sign.

To sum up what we have observed:

x^n-y^n is divisible by $x-y$ whatever n is,

and likewise by $x+y$ when n is even.

x^n+y^n is divisible by $x+y$ when n is odd,

but is not divisible by $x-y$, whatever n is; understanding by "divisible," divisible without a remainder, that is, that the denominator is a *factor* of the numerator; and understanding by n always a positive integer.

Another way of putting it is as follows:

$x-y$ and $x+y$ are both factors of x^n-y^n if n is even,
$x-y$ only is a factor of x^n-y^n if n is odd,
$x+y$ only is a factor of x^n+y^n if n is odd,
neither is a factor of x^n+y^n if n is even.

Or thus, which forms an easy way of remembering the facts:

x^3+y^3 is divisible by $x+y$,
x^3-y^3 is divisible by $x-y$,
x^2-y^2 is divisible by both,
x^2+y^2 is divisible by neither.

General expression for any odd number.

It is handy to be able to discriminate between an odd and an even number algebraically.

It is done thus:

$2n$ is always even, if n is an integer.

$2n \pm 1$ is always odd, again if n is an integer.

The nth odd number is $2n-1$ (hence this is commonly the expression used for an odd number);

e.g. 5 is the third odd number and is equal to $(2 \times 3)-1$; 11 is the sixth odd number and is equal to $(2 \times 6)-1$; and so on.

The hundredth odd number is therefore 199 and the 365th is 729.

Arithmetical Progression.

Now take some examples of A.P.

An interesting case is to find the sum of the first n consecutive n odd numbers added together, that is to find the value of

$$1+3+5+\ldots+(2n-1).$$

This sum might be found by experiment, thus:

$$1+3 = 4 = 2^2,$$
$$1+3+5 = 9 = 3^2,$$
$$1+3+5+7 = 16 = 4^2.$$

So the sum of the first four odd numbers is 4^2 and of the first five will be found to be $5^2=25$.

Trying a few more we come to the experimental conclusion that the sum of the first n odd numbers will be n^2.

$$1 + 3 + 5 + 7 + \ldots + 2n - 1 = n^2.$$

The annexed diagram illustrates to the eye the facts that

$$1 + 3 = \text{the square of } 2,$$
$$1 + 3 + 5 = \text{the square of } 3,$$
$$1 + 3 + 5 + 7 = \text{the square of } 4,$$

and so on.

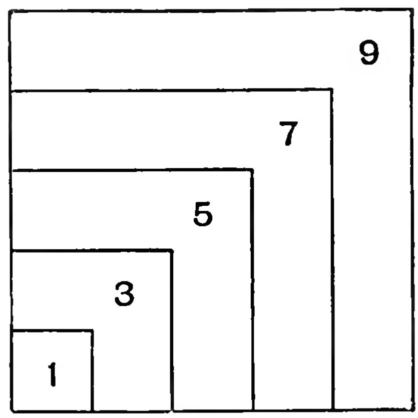


FIG. 34.

The sum of the first n *natural* numbers is not so simple, but is a good problem to solve experimentally, thus:

$$1 + 2 = 3,$$
$$1 + 2 + 3 = 6,$$
$$1 + 2 + 3 + 4 = 10.$$

So we have a series, rather a notable series, with differences increasing by 1 each time.

For the sum is

one term	two terms	three terms	four terms	five terms	six terms
1	3	6	10	15	21, etc.
			2×5	$2\frac{1}{2} \times 6$	3×7

What would be the number thus to be placed under n terms?

The answer is $\frac{1}{2}n(n+1)$; and it is possible but not quite easy to guess that.

It is worth remembering however: the sum of the first n natural numbers is $\frac{1}{2}n(n+1)$.

As for the sum of the even numbers, that is a very simple matter, for it is merely double the preceding; or it may be regarded as the sum of the odd numbers plus n;

$$2 + 4 + 6 + 8 + 10 \ldots + 2n = n^2 + n$$
$$= n(n+1).$$

This method of guessing and verifying will not carry us far: a reasoned process of arriving at a result is far more powerful and effective. Algebra enables us to reason things out; and the customary method for the sum of an A.P. is as follows:

Let the general arithmetical progression be the following, to n terms,

$$a, \qquad a+b, \qquad a+2b, \qquad a+3b \ldots a+(n-1)b;$$

write it again, but backwards,

$$a+(n-1)b, \ a+(n-2)b, \ a+(n-3)b, \ \ldots\ldots, \ a.$$

Now add the two series together, term by term, as they stand one under the other; and the result will be $2a+(n-1)b$ every time.

Hence, since there are n terms, the result of the double series added together, if S is the sum of a single series, will be

$$2S = n\{2a+(n-1)b\};$$
$$\therefore\ S = na+\tfrac{1}{2}n(n-1)b.$$

This is the general result for an A.P.

For example, to test it by special cases:

In the case of the first n natural numbers $a = b = 1$, and so

$$S = n+\tfrac{1}{2}n(n-1)$$
$$= \tfrac{1}{2}n(n+1),$$

as we have already found by experiment.

In the case of the first n odd numbers, a is 1 and $b = 2$;

$$S = n+n(n-1) = n^2,$$

as we also found experimentally.

It is very instructive and pleasing to see how a general formula thus gives special cases, and it is one of the verifications by which a general formula should always be tested.

The following is interesting for practice:

$$1 = 1^3$$
$$3+5 = 2^3$$
$$7+9+11 = 3^3$$
$$13+15+17+19 = 4^3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots \text{etc.}\ldots$$

Other Series.

The number of series or progressions that can be dealt with is enormous, is indeed infinite; and is too large a subject for us to enter upon in this book. Suffice it to say that many others occur in practice besides the simple ones which are best known.

This series, for instance,

$$1^2+2^2+3^2+4^2$$

is neither a geometric nor an arithmetic nor a harmonic progression. Something like it occurs in the overtone frequencies of vibration of plates and bars.

Manifestly we might have

$$1^2,\ 3^2,\ 5^2,\ \ldots,$$

or

$$1^3,\ 2^3,\ 4^3,\ 8^3,$$

and so on; any number of such series could be invented.

There is one simple series that we came across recently on page 331, the difference of whose terms was constantly and steadily increasing: the series 1, 3, 6, 10, 15, etc.

If we started with this series and took the differences we should get an A.P. series, and this is a process we might continue; thus:

Start with this,

0 1 3 6 10 15 21 28 ....

Take differences,

1 2 3 4 5 6 7 ... an A.P.

Take second differences,

1 1 1 1 1 1 a series of constants.

Take third differences,

0 0 0 0 0 a series of zeros.

Suppose we start with a different series, say the natural series of square numbers,

0, 1, 4, 9, 16, 25,

the differences of these will give the series of odd numbers; while the second differences would be constant.

If we took any geometrical series the differences would be the same series again, multiplied by a factor, the factor being one less than the common ratio.

Hence the differences of the powers of 2, viz. 1, 2, 4, 8, 16, 32, would be the same series over again.

The binomial coefficients can be obtained by interjecting a single 1 into the middle of a row of noughts and then adding adjacent terms to make a term of the next series, as thus,

0	0	0	0	0	1	0	0	0	0	0		
0	0	0	0	1	1	0	0	0	0			
0	0	0	1	2	1	0	0	0				
0	0	1	3	3	1	0	0					
0	1	4	6	4	1	0						
1	5	10	10	5	1							
1	6	15	20	15	6	1						
1	7	21	35	35	21	7	1					
1	8	28	56	70	56	28	8	1				
1	9	36	84	126	126	84	36	9	1			
1	10	45	120	210	252	210	120	45	10	1		
1	11	55	165	330	462	462	330	165	55	11	1	
1	12	66	220	495	792	924	792	495	220	66	12	1

The simplest illustration of an arithmetical progression is the natural series of numbers—the ordinary counting of a child. The most important instance of an arithmetical progression that occurs in nature is afforded by *time*. It is true that it progresses continuously and not by jerks, but the motion of a clock hand is a jerky motion, and the succession of days, weeks, and years divide the continuum into units for measuring purposes, and represent a perfectly uniform and inexorable constant rate of progress.

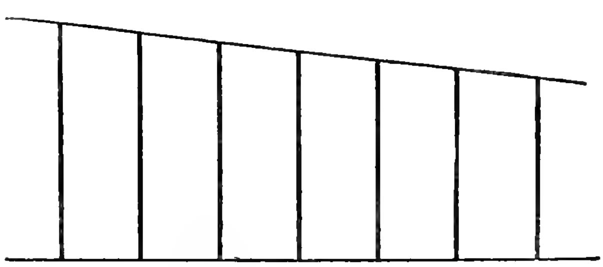
A set of numbers are said to be in geometrical progression when their logarithms are in arithmetical progression. The notes of a piano are in this predicament, when estimated by their vibration frequencies. The chromatic scale, on a tempered instrument like a piano, proceeds by equal musical intervals, but these intervals are characterised by equal *ratios* of vibration frequency, every octave having double the vibration rate of its predecessor; in other words the factor 2 carries us over the interval of an octave, the factor $\frac{3}{2}$ gives successive fifths, and so on; so that the same musical interval, in different parts of the register, is characterised by a constant difference of logarithm.

A set of numbers are said to be in harmonic progression when their reciprocals are in arithmetical progression.

The series of square numbers have their *roots* proceeding in A.P.; another series we have encountered has consecutive *differences* in A.P.; another series might have successive *ratios* in A.P.; and so on.

Geometrical Illustrations.

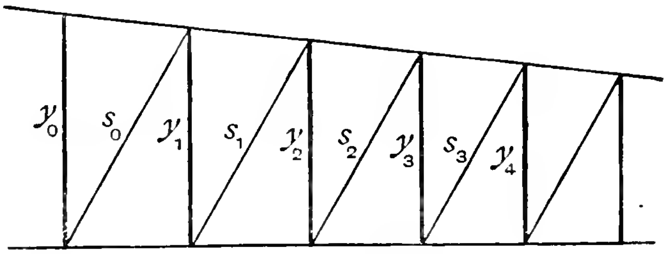
The heights of a row of palings may be used to illustrate the three best known modes of progression, if their tops all reach a sloping straight line. If they are spaced simply at equal intervals, they of course form an A.P.; if they are spaced so that lines drawn from the foot of each to the top of the next are all parallel, they will form a G.P.; but if spaced so that a line from the foot of each to the top of the next-but-one bisects the intermediate one, they form an H.P. The three figures annexed illustrate this.



Arithmetical
FIG. 35.

Concerning fig. 35 there is nothing to be said but what is obvious.

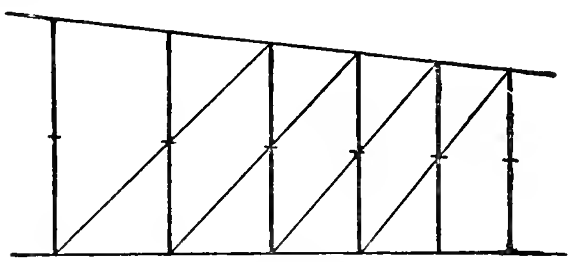
Concerning fig. 36 it can be pointed out that the triangles formed are all similar, that the lengths of the slant lines are in G.P. as well as the vertical lines; and so are the areas of the



Geometrical
FIG. 36.

triangles. They may be said to illustrate the successive heights attained by a bouncing ball: which heights are also in G.P.

Fig. 37 is the most notable; it may be regarded as the perspective view of a series of equal rectangles or parallelograms—the perspective view in fact of a uniform fence. Hence it is useful in drawing *metrical* perspective figures.



Harmonic
FIG. 37.

Proofs.—The proof that fig. 36 represents a geometrical progression is almost obvious, since by construction the triangles are similar, their sides being parallel; hence

$$\frac{y_0}{y_1} = \frac{s_0}{s_1} = \frac{y_1}{y_2} = \frac{s_1}{s_2} = \frac{y_2}{y_3}, \text{ etc.}$$

The fact that fig. 37 gives a harmonic progression can be established thus:

Let a, b, c be three verticals erected so that a line from the foot of a to the top of c, or from the foot of c to the top of a, bisects the intermediate height b, which therefore divides the base in some ratio $m:n$, then it can be shown that b is the harmonic mean of a and c; for by similar triangles

$$\frac{\frac{1}{2}b}{a}=\frac{n}{m+n}, \text{ and } \frac{\frac{1}{2}b}{c}=\frac{m}{m+n};$$

therefore $$\frac{\frac{1}{2}b}{a}+\frac{\frac{1}{2}b}{c}=1$$

or $$\frac{1}{a}+\frac{1}{c}=\frac{2}{b};$$

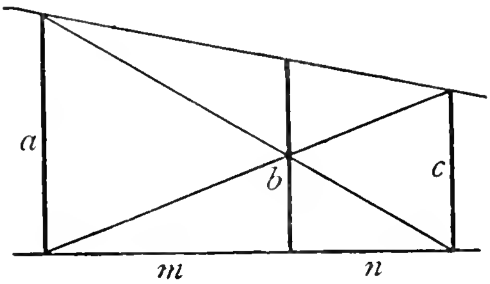


FIG. 38.

wherefore the reciprocals are in arithmetical progression.

In fig. 37 it is convenient to call the slant lines transversals, and to say that the transversal from the foot of each passes through the mid-point of the next to the top of the next-but-one.

Another geometrical illustration of a G.P. is the following—

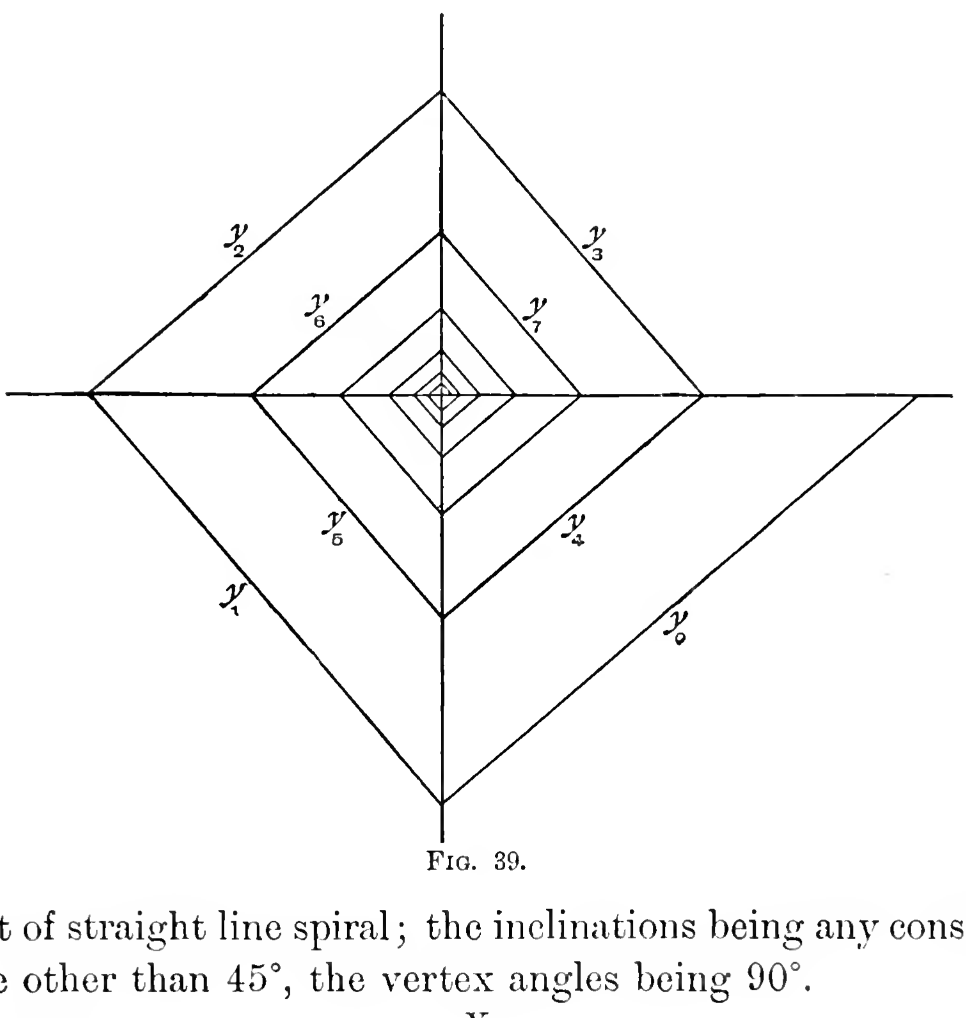


FIG. 39.

a sort of straight line spiral; the inclinations being any constant angle other than 45°, the vertex angles being 90°.

Successive sides of the spiral are in G.P., and so are the distances of successive vertices from the centre.

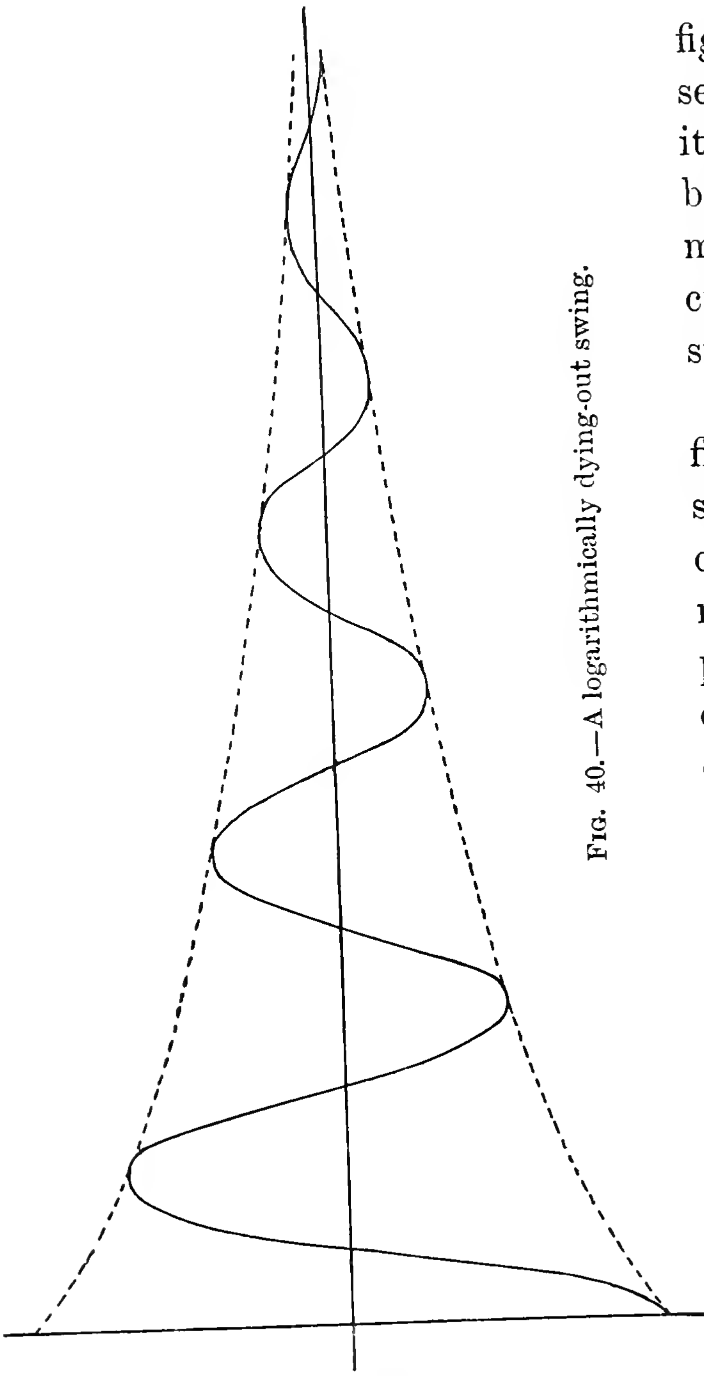


FIG. 40.—A logarithmically dying-out swing.

To convert fig. 35 or fig. 41 into the representation of a G.P. as it stands, the roof must be made of a logarithmic or exponential curve instead of a straight line.

Thus fig. 9 and fig. 47 already represent a G.P.; each vertical height is the Geometric Mean of any pair of heights equidistant on either side of it.

The 'amplitudes' of the swings of a dying-out pendulum constitute a G.P.: the 'periods' of successive swings constitute an A.P. See (fig. 40.)

The temperatures of a cooling body, read every minute, constitute an approximate G.P., and if plotted would give a logarithmic curve: looking like fig. 9 or fig. 78, or the dotted line in fig. 40, or part of the figure on page 179.

CHAPTER XXXVI.

Means.

A THING of some interest and use is the mean or average of a set or a pair of terms in a progression. In an A.P. the mean can be found by adding and halving the two extreme terms. Thus for instance in the progression

$$7 \quad 9 \quad 11 \quad 13 \quad 15$$

11 is the mean term, and it can be found as the half sum of 9 and 13, or the half sum of 7 and 15.

The **arithmetic mean** of a and c is $\frac{1}{2}(a+c)$; for calling this b, it makes $b-a = c-b$, that is, it gives a common difference in the progression a, b, c; and it is illustrated by the figure, where b is the mean height of the trapezium shown, whose area is therefore b times the base.

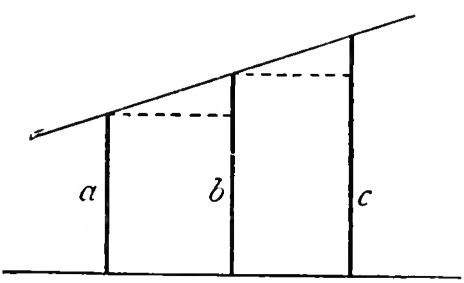


FIG. 41.

The arithmetic mean of	1	and	7	is	4
of	0	and	100	is	50
of	0	and	9	is	$4\frac{1}{2}$
of	-1	and	$+1$	is	0
of	-6	and	$+8$	is	1
of	-3	and	$+9$	is	$+3$
of	-9	and	$+3$	is	-3
of	12	and	90	is	51

because $51 - 12 = 39$, and $90 - 51 = 39$; or because

$$51 = 6 + 45.$$

In general then the **arithmetic mean** is the half sum, the sum being understood as the algebraic sum, paying attention to sign.

The **geometric mean** of two terms is the square root of their product, because this would give a common ratio; thus if three terms a, b, c are in G.P., b must equal $\sqrt{(ac)}$, because

$$\sqrt{(ac)} : a = c : \sqrt{(ac)} = \sqrt{\left(\frac{c}{a}\right)}, \text{ the common ratio;}$$

in other words the common ratio $\frac{b}{a}$ or $\frac{c}{b}$ must be equal to $\sqrt{\frac{c}{a}}$.

In the progression a, ar, ar^2 the middle term is plainly the square root of the product of the end terms.

The Geometric Mean is also called a "mean proportional."

To illustrate a geometric mean it is customary to use either a right-angled triangle or a circle. Thus if the two lengths whose geometric mean is required are CA and CB, any circle drawn through A and B has the property that its tangent drawn from C is equal to the geometric mean required; for by Euclid III. 36, $CP^2 = CA . CB$; hence incidentally we arrive at the proposition that all the circles that can be drawn through the two points A and B can be cut at right angles by a certain circle drawn round C as centre; because the length CP is constant. If only the points A and B had been initially given, then a number of such points C could be found, each with its appropriate length of radius, by drawing a tangent, or a series of tangents, to any one of the circles.

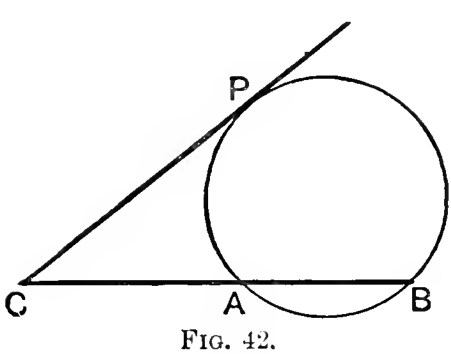


FIG. 42.

If a right-angled triangle ABC be drawn, and a perpendicular be let fall from the right angle C on to the opposite side, the length of this perpendicular is a mean proportional between the segments of the base:

$$CD = \sqrt{AD \,.\, DB}, \text{ since } \frac{AD}{CD} = \frac{CD}{BD}.$$

Similarly AC is a mean proportional between AB and AD,
and BC ,, ,, ,, ,, BA and BD.

The same thing is true for a semicircle, since the angle in a semicircle is a right angle. Euclid III. 31.

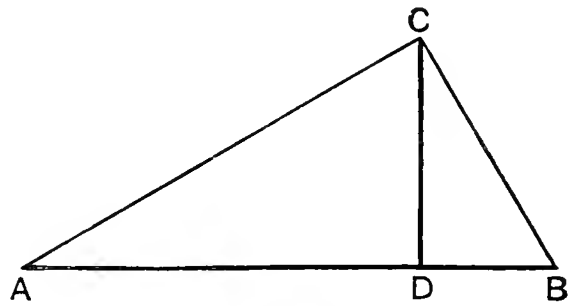


FIG. 43.

Hence an easy construction for finding the Geometric Mean of two lengths is to place them end to end, as AD, DB, construct a semicircle on the whole length AB thus compounded, and erect a perpendicular DC at the junction point of the two lengths. This is the G.M. required. (Cf. p. 274.)

Or if the two given lengths had been AB and AD, then the distance AC would be their G.M.

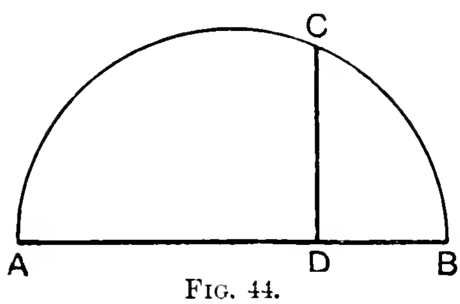


FIG. 44.

The **harmonic mean** of two terms is such that it would be the arithmetic mean of the two terms inverted.

For instance, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ are in H.P., and $\frac{1}{3}$ is the harmonic mean between $\frac{1}{2}$ and $\frac{1}{4}$.

Let a, b, c be in harmonic progression, then $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in arithmetical progression, and $\frac{1}{b} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right)$;

wherefore $$b = \frac{2ac}{a+c}.$$

The harmonic mean can therefore be described as twice the product divided by the sum.

Geometrically it could be represented by setting up the two given numbers as parallel measured lengths, like a and c, and joining their ends both direct and crosswise.

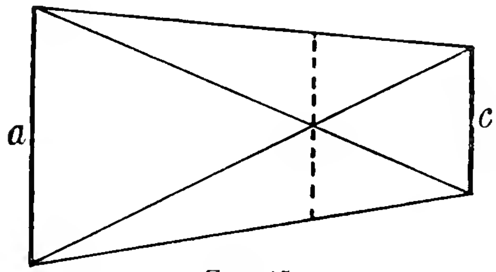


FIG. 45.

Then the parallel drawn through the crossing point is the harmonic mean of a and c. It is represented by a dotted line in the figure.

The proof of this construction is given above in connexion with figs. 37 and 38.

If the two outer lines of the figure are continued till they meet, the fourth position thus determined forms a H.P. with the three positions determined by the crossings depicted in the figure; and they are familiar in elementary geometrical optics.

Examples.

The student should cover up the right-hand column and reckon the entries in it. They are all intended to be done in the head.

The *geometric mean*	of	1	and	9	is	3.
	of	9	and	9	is	9.
	of	1	and	100	is	10.
	of	0	and	100	is	0.
	of	0	and	n	is	0.
	of	9	and	36	is	18.
	of	4	and	25	is	10,
or generally,	of	m^2	and	n^2	is	mn.
	of	40·5	and	24·5	is	31·5,
that is to say	of	$\frac{1}{2}m^2$	and	$\frac{1}{2}n^2$	is	$\frac{1}{2}mn$.
	of	$\frac{1}{4}$	and	81	is	4·5.
	of	7	and	$\frac{1}{7}$	is	1.

the G.M.	of	$\frac{1}{9}$	and	16	is	$1\cdot\dot{3}$.
	of	$\frac{1}{7}$	and	252	is	6.
	of	147	and	$\frac{1}{27}$	is	$2\cdot\dot{3}$,
	of	49	and	$\frac{1}{9}$	is	$2\cdot\dot{3}$.
or generally,	of	am^2	and	$\frac{1}{an^2}$	is	$\frac{m}{n}$.
	of	am	and	$\frac{a}{m}$	is	a,
e.g.	of	72	and	8	is	24.
	of	-16	and	-1	is	-4.
	of	-16	and	-100	is	-40.
	of	$-\frac{1}{2}$	and	-72	is	-6.
	of	-14	and	-7	is	$-7\sqrt{2}$.
	of	ab	and	a	is	$a\sqrt{b}$.

To find the *harmonic mean* b of two terms a and c, we can write down that
$$\frac{1}{a}-\frac{1}{b}=\frac{1}{b}-\frac{1}{c},$$
or that
$$\frac{b-a}{ab}=\frac{c-b}{bc},$$
or that
$$\frac{b-a}{c-b}=\frac{a}{c}.$$

The two most usual expressions for a three-term harmonic progression are
$$\frac{1}{a}+\frac{1}{c}=\frac{2}{b},$$
and
$$b=\frac{2ac}{a+c};$$
these are simple and useful, and we come across them constantly in elementary optics. In reflexions from a small curved mirror the object and image and the centre of curvature of the mirror are situated at points whose distances from the mirror itself run in a harmonic progression.

So also in lenses, if a is the distance of object from the lens, and c the distance of the image, $\frac{1}{2}b$ is its focal length.

This is set down here merely as a reminder.

Illustrations.—The harmonic mean of $\frac{1}{4}$ and $\frac{1}{6}$ must be $\frac{1}{5}$; and twice the product by the sum of the two numbers is

$$\frac{2 \times \frac{1}{4} \times \frac{1}{6}}{\frac{1}{4} + \frac{1}{6}} = \frac{\frac{1}{12}}{\frac{10}{24}} = \frac{24}{120} = \frac{1}{5}.$$

The harmonic mean of 4 and 6 is

$$\frac{2 \times 4 \times 6}{4 + 6} = \frac{48}{10} = 4 \cdot 8.$$

The harmonic mean of 1 and 99 $= \dfrac{2 \times 99}{100} = 1 \cdot 98.$

The harmonic mean of 0 and 1 is 0.

The harmonic mean of 17 and 13 is

$$\frac{34 \times 13}{30} = \frac{442}{30} = 14 \cdot 7\dot{3}.$$

The harmonic mean of -2 and $+6 = \dfrac{-4 \times 6}{6 - 2} = \dfrac{-24}{+4} = -6.$

" " of -2 and $+2 = -\infty$.

" " of -5 and $-7 = \dfrac{70}{-12} = -5\frac{5}{6}$.

" " of -1 and $-9 = \dfrac{18}{-10} = -1 \cdot 8$.

" " of 1 and $\infty = 2$.

The geometric mean of 5 and 20 is $\sqrt{5 \times 20} = 10$, the common ratio being 2.

The geometric mean of 1 and 16 is 4.

" " of 2 and 32 is 8.

" " of 4 and 9 is 6.

" " of 8 and 2 is 4.

" " of -8 and -2 is -4.

" " of -8 and $+2$ is imaginary.

" " of -1 and -9 is -3.

" " of a and b is $\sqrt{(ab)}$.

Comparing the three means of the same two quantities a and b,

the arithmetic mean is $\frac{1}{2}(a+b)$ and is the biggest of the three,
the geometric mean is $\sqrt{ab}$,
the harmonic mean is $\dfrac{2ab}{a+b}$ and is the smallest of the three.

The H.M. may be considered as $\dfrac{(\text{G.M.})^2}{\text{A.M.}}$, that is the ratio of the square of the G.M. to the A.M.; or it is equal to the G.M. multiplied by the proper fraction G.M./A.M.

Examples.

Take any two numbers, say 4 and 9 :

the arithmetic mean is 6·5,
the geometric mean is 6·0,
the harmonic mean is $\frac{72}{13}$ = 5·53846....

Let the two numbers be 1 and 25 :

the A.M. is 13,
the G.M. is 5,
the H.M. is $\frac{25}{13}$ = 1·923....

Let the two numbers be 49 and 36 :

the A.M. is 42·5,
the G.M. is 42·0,
the H.M. is $\dfrac{(42)^2}{42{\cdot}5} = \dfrac{42}{42{\cdot}5} \times 42$,

which is necessarily less than the G.M.

Let the two numbers be 0 and 4 :

the A.M. is 2,
the G.M. is 0,
the H.M. is 0,

but not the same 0, it is half the G.M squared.

Let the two numbers be 6 and -6;

the A.M. is 0 :
the G.M. is imaginary,
the H.M. is $-\infty$.

Let the two numbers be -3 and -6:

the A.M. is $-4\cdot5$,

the G.M. is $-3\sqrt{2} = -4\cdot2426$,

the H.M. is $-4\cdot0$.

In some cases the order of *numerical* magnitude is inverted; but, when compared with positive, the smallest negative quantity is represented by the largest number. If heights of mountains were reckoned from sky instead of from earth, as by dropping a plummet from a balloon, the lowest mountain would need the longest plumb-line to reach it. The lowest parts of the solid earth are beneath the sea and require a long sounding line to reach them.

The A.M. is of course always half way between the two numbers.

The G.M. is nearer to the smaller one, and ultimately coincident with the smaller one when the other is infinitely bigger.

The H.M. is less than the G.M. in the ratio of $\frac{\text{G.M.}}{\text{A.M.}}$; or $\text{H.M.} \times \text{A.M.} = \text{G.M.}^2$, or the G.M. is a mean proportional between the other two.

Mean or Average of a Number of Terms.

In taking a mean of terms there is no need whatever that those terms should form any sort of progression or ordered series. Hitherto we have only taken the mean of two terms, and two terms cannot possibly determine any kind of progression, any more than two points can determine a curve. But we can reckon the arithmetical or the geometrical mean of any number of terms as follows:

Suppose we want the mean of a set of observations of temperature, taken at every hour of the day, so as to determine the mean temperature during the day of 12 hours, say from

8 a.m. to 8 p.m. Let the thermometer readings be the following—there will be 13 readings, because of the beginning and end points of time between which the twelve hours lie:

60·5
65·0
67·2
67·3
71·25
75·0
79·0
77·0
74·6
70·3
62·4
55·3
53·7

13 | 878·5
67·58

Add them up and divide by the number of them, that is by 13. This is the mean or average of the readings, and is found to be 67·58. It is apparently a summer day with a warm and probably cloudy morning giving place to a clearer sky and cooler evening.

If the temperature readings were plotted and joined, the result would be a curve (fig. 46); and the average height of this curve would be the mean temperature.

The average height must be approximately 67·58; but when the curve is drawn by a recording thermometer, so as to give the temperature not only at every hour, but at every instant, a more exact determination of the average can be made.

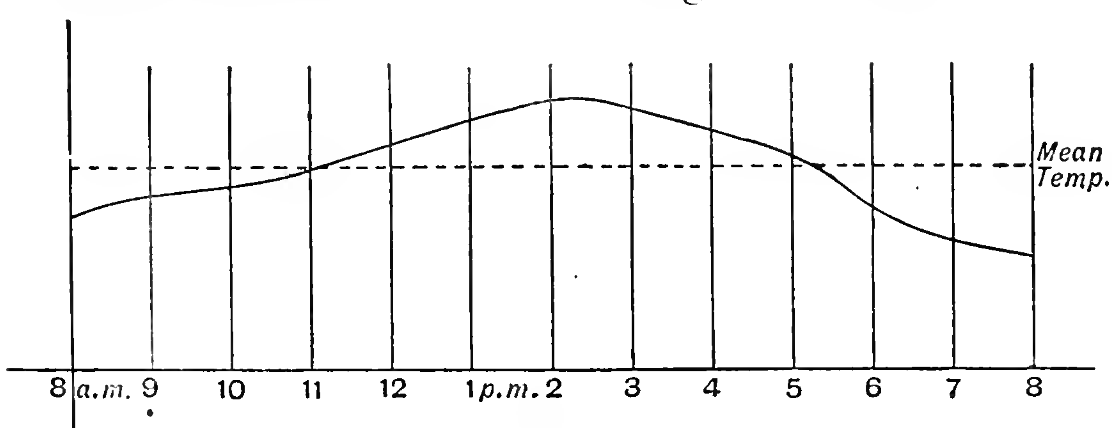


Fig. 46.

The average or mean height of such a curve is the height of a rectangle with the same base which shall equal the curve in area, as shown in the figure by the dotted line.

One way of finding out the area of such a curve, and in that way of obtaining its mean height, is to cut the above figure out in cardboard or tinfoil or sheet lead, and then weigh it;

weighing at the same time a rectangle or square of known dimensions cut out of the same sheet.

Thus suppose the curve carefully drawn on such a uniform sheet, on a scale which gave 1 horizontal centimetre to each hour and 1 vertical millimetre to each degree, and that the figure bounded as above was carefully cut out and found to weigh 4·98 grammes; while a rectangle of the same base 12 centimetres, and height 7 centimetres, was found to weigh 5·16 grammes.

We should know that the area of the curve-bounded figure was $\frac{4·98}{5·16} \times 7 \times 12$ sq. centimetres, and that its average height was $\frac{4·98}{5·16} \times 70$ millimetres.

This process would give the average height, and therefore the number of degrees in the average temperature, as 67·56; and, if carefully carried out, it should be more correct than merely averaging numerical readings taken each hour, for it averages the temperature recorded from instant to instant.

To follow a process of this kind profitably, the best plan is actually to do it, and then the method of working will naturally occur to you with a little thought; and a good result can be obtained with some handicraft skill. It is a practical method of experimentally performing the operation known as **integrating**; it is integration between definite limits, or the finding of a definite integral of a function represented by a curve.

Weighted Mean.

Generally the word "mean" implies the simple arithmetic mean, and the mean of several numbers, say n_1, n_2 and n_3, is $\frac{n_1 + n_2 + n_3}{3}$, which is often written $\bar{n}$ and read "n bar."

The average of $a_1, a_2 \ldots a_n$ is $\bar{a} = \frac{1}{n}(a_1 + a_2 + \ldots + a_n)$; but sometimes we have to do with a *weighted* mean. One case is when a series of observations of the same thing are taken under different circumstances, and some of the circumstances are more favourable than others; for instance if the height of a flagstaff were being measured by the length of its shadow, at noon on successive days at the time of the summer solstice, and suppose the record of the shadow measurements was entered thus:

15·46 feet	day fair	observer W. Smith.
15·30 ,,	cloudy	,, ,,
15·47 ,,	day bright	observer E. Jones.
15·50 ,,	weather hazy	,, ,,
25·6 ,,	day bright	observer J. Williams,

and the most probable result were required.

First of all the last observation would have to be thrown aside altogether, because Williams has evidently made a mistake in the very first significant figure, and although his observation may be correct in the last figure and almost certainly means 15·6, it is hardly safe to begin doctoring results; it is safer to reject any that thus show obvious signs of carelessness.

The other observations may have different weights attached to them, and to know how to attach weights satisfactorily needs considerable experience, and experience too with these same observers, because it may happen that Jones is known to be a more trustworthy and exact observer than Smith; each person has what is called a "personal equation," some always tend to read slightly too large, others slightly too small, while others cannot be trusted to more than say three significant figures.

Let us suppose that an experienced person decides to attach the weight 3 to the first observation, the weight 1 to the second because of the clouds, the weight 6 to the next because Jones is a good observer, and the weight 2 to the next because of the haze, the weighted mean of the set of observations would be obtained as follows, attending only to the decimal places because the 15 is common to all:

$$\frac{(3 \times \cdot 46) + (1 + \cdot 30) + (6 \times \cdot 47) + (2 \times \cdot 50)}{3 + 1 + 6 + 2}$$

$$= \frac{1\cdot 38 + \cdot 30 + 2\cdot 82 + 1\cdot 00}{12} = \frac{5\cdot 50}{12} = \cdot 4583;$$

wherefore the result as thus determined would be given as 15·458, or say 15·46 feet to four significant figures: some probable error affecting the last place.

General Average.

An average in general may be better expressed thus:

Let n_1 observations give a result x_1
let n_2 „ „ „ x_2
$\vdots$ $\vdots$
n_6 „ „ „ x_6,

then the total number of observations is $n_1 + n_2 + \ldots n_6$, and the appropriate weighted mean or average of the whole number of observations,

$$\bar{x} = \frac{n_1 x_1 + n_2 x_2 + \ldots n_6 x_6}{n_1 + n_2 + \ldots n_6},$$

commonly written as $\bar{x} = \dfrac{\Sigma(nx)}{\Sigma(n)}$,

the Σ being read "sum."

This is a most important and commonly occurring form of average, or arithmetic mean, of any number of like and unlike quantities.

Geometric Mean of several Numbers.

To find the G.M. of two quantities we multiply them together and extract the square root. Similarly to find the G.M. of three quantities we should multiply them together and extract the cube root; and to find the G.M. of say six quantities, multiply them all and extract the sixth root. But this would have to be done by logarithms; so the process is better put into logarithmic form from the first.

To find the geometric mean of n quantities

$$a_1\, a_2 \,\ldots\, a_n.$$

Find the arithmetic mean of their logarithms:

$$\log a = \frac{1}{n}(\log a_1 + \log a_2 + \ldots \log a_n),$$

the resulting a as thus calculated from its logarithm will be the G.M. required.

Example.—To find the G.M. of

92, 100, 121, and 89;

look out their logarithms, add, and divide by four,

$$\begin{array}{r|l} & 1{\cdot}9638 \\ & 2{\cdot}0000 \\ & 2{\cdot}0828 \\ & 1{\cdot}9494 \\ \hline 4 & 7{\cdot}9960 \\ \hline & 1{\cdot}9990 = \log \text{ of } 99{\cdot}8 \end{array}$$

Therefore the G.M. required is 99·8.

The A.M. would have been 100·5.

As to the H.M. of more than two quantities I do not remember that it is often required: it can be got by taking the arithmetic mean of the reciprocals of the given set of numbers and then the reciprocal of that.

CHAPTER XXXVII.

Examples of the practical occurrence of Progressions in Nature or Art.

In illustrating a subject by examples there is a great advantage in selecting natural examples in place of artificial ones. Natural examples may not be quite so easy as artificial ones, but they are vastly better worth studying. Artificial examples are often easy and handy for practice, and many of them can be done in a short time, but a real or naturally occurring example will take you into the essence of the subject and is worth dwelling on long and steadily. Every such example is more than an example: it opens a chapter, and sometimes needs a treatise.

The chief instance of the natural occurrence of a geometrical progression is in the theory of "leaks"—a leak of steam out of a boiler, or of compressed air or water out of a reservoir, of heat out of a cooling body, of electricity out of a charged conductor; these and many other instances are all subject to the same mathematical law—the law of a decreasing geometric progression. See Chapters XL. to XLII.

A commercial example of the occurrence of G.P. is the institution known as compound interest, when money invested in some business undertaking is allowed to supply the necessities and the supplementary accessories of effective human labour; on the strength of which, under good management, the sum invested increases in value at an ascertainable or arbitrarily specified rate, and may accumulate until it becomes a very large fortune.

Let us take this case first, for it may perhaps seem simpler than an example from Physics.

Interest.

Suppose £1000, invested in machinery and wages, enables a workman to produce fifty pounds worth of goods every year more than need be expended on advertising the goods, carrying them to their destination, feeding and clothing the workman, patching up his shed and repairing the machinery; it is called capital, and is said to increase at the rate of 5 per cent. per annum, the increase being called interest. If the fifty pounds is taken away and otherwise utilised, so that the original capital of £1000 remains what it was, without increase or diminution as the years go by, it is called simple interest, and is an example of arithmetical progression. But if the fifty pounds is invested in improved machinery and in extra assistance, in such a way that it too brings in a profit at the same rate; and if this is steadily done each year as interest accumulates, so that it is always added to the capital, which thus goes on increasing; it is called compound interest, and is governed by the law of geometrical progression.

Every year the capital is increased in the ratio of $\frac{105}{100}$, or every pound at the beginning of the year becomes a guinea at the end of it. If this is supposed to go steadily on, what will be the result after say 15 years?

The result will be that the original capital has been increased $\frac{21}{20}$ths in one year, and this new capital has been increased $\frac{21}{20}$ths in the second year, or the original capital

$\frac{21}{20} \times \frac{21}{20}$ in two years. In three years the original capital will have increased in the ratio $\left(\frac{21}{20}\right)^3$, and in 15 years $\left(\frac{21}{20}\right)^{15}$.

Now $\left(1+\frac{1}{20}\right)^{15}$ could be found by the binomial theorem if we liked: and we know that as a first approximation it will be $1+\frac{15}{20}$.

Introducing the second order approximation

$$1+nx+\frac{n \,.\, n-1}{\underline{|2}}\,x^2,$$

we shall find its value to this order of approximation as

$$1+\frac{15}{20}+\frac{15\times 14}{2}\cdot\frac{1}{400} = 1{\cdot}75+\frac{7\times 15}{400}$$

$$= 1+\frac{3}{4}+\frac{105}{400} = 2\tfrac{1}{80},$$

which equals approximately 2; that is to say the capital in 15 years will be by this operation a little more than doubled, and will have become rather more than £2000.

Now a quantity which is frequently doubled becomes, as we know, extraordinarily large after the operation has been repeated several times. If it doubles every 15 years, in 60 years it will have become £16000, and in a century and a half it will have to be multiplied by 2^{10}, viz. 1024; that is to say the capital will have swelled to more than a million.

A "penny" put out to 5 per cent. compound interest in the time of Cæsar would now theoretically far exceed all the material wealth of the world in value.

But though an approximate calculation of compound interest is instructive, there is no need to make it approximate, we can calculate it exactly if we choose. We have only to raise the ratio of the G.P., viz. 1·05 or $1\frac{1}{20}$, to the 15th power in order to find the value after 15 years.

This we should naturally do by logarithms,

$$\log 1{\cdot}05 = {\cdot}0211893,$$

$$15 \times \log 1{\cdot}05 = {\cdot}3178365 = \log \text{ of } 2{\cdot}078914.$$

So the original capital of £1000 becomes increased to £2078. 18s. 3d. in fifteen years, at five per cent. per annum compound interest.

Suppose we wished to find at what rate of compound interest a sum of money would exactly double itself in any given time, say for instance in 12 years, we should have to proceed thus:

Let x be the rate of interest,

then $(1+x)^{12}$ has to equal 2,

wherefore $1+x = \sqrt[12]{2}$.

We must use logarithms to find the twelfth root of 2.

$$\log 2 = {\cdot}301030,$$

$$\tfrac{1}{12}\log 2 = {\cdot}025086 = \log \text{ of } 1{\cdot}05945,$$

which equals $1+x$, wherefore $x = {\cdot}05945$,

or the rate of interest must be 5·945, or nearly six, per cent., in order that doubling may occur every dozen years.

Let us see at what rate doubling will occur in thirty years.

$$\tfrac{1}{30}\log 2 = {\cdot}0100343$$

$$= \log \text{ of } 1{\cdot}0241,$$

or a little more than 2·4 per cent., about £2. 8s. 2d. added to every hundred pounds per year.

Any rate of interest will double property if sufficient time be allowed to it; but if we wanted to double capital every six years, we should need a high rate of interest:

$$(1+x)^6 = 2,$$

$$\tfrac{1}{6}\log 2 = {\cdot}050172 = \log \text{ of } 1{\cdot}12247,$$

indicating about $12\frac{1}{4}$ per cent.

Approximately therefore the doubling time and the rate of interest vary inversely. If one is increased, the other can be decreased in roughly something like the same proportion, especially when the rate is small.

That is apparent at once if we expand by the binomial and put it to double itself in n years:

$$(1+x)^n = 1+nx \text{ approximately } = 2, \text{ say,}$$

wherefore $\quad nx = 1$ approximately,

or n and x vary inversely as one another, to a first approximation. (See Chap. XXXIX.)

But then this first approximation is exactly *simple* interest, it ignores the x^2 and x^3 and higher terms; and it is just in the presence of those terms that the virtue of *compound* interest consists.

If x is added to each pound every year, but the interest is not allowed to become part of the capital, so as to increase, the amount becomes at the end of n years simply $(1+nx)$ times its original value. Thus at 5 per cent. simple interest, so that $x = \cdot 05$, £1000 in 1 year will become 1·05 times £1000, or £1050. In three years it will be £1150, and in 20 years it will be multiplied by $1+(20 \times \cdot 05) = 2$; that is to say it will be just exactly doubled in twenty years.

So whereas compound interest at 5 per cent. doubles an amount in about 15 years, simple interest does the same thing in 20 years; and the interest could have been drawn and otherwise utilised all the time.

At 10 per cent. simple interest a sum would be doubled in ten years, and at 1 per cent. simple interest a sum would be doubled in a century; whereas at 1 per cent. compound interest, in a century, it would have increased in the proportion $(1\cdot 01)^{100} = 2\cdot 705$, that is would have considerably more than doubled, though it would not have trebled.

The advantage of compound interest over simple interest tells more at high rates, for then the higher powers in the expansion become important; and therein lies the difference. It is instructive to plot the two things. Simple interest increase would be represented by a straight line law, compound interest by an exponential curve, the two starting off together, but ultimately separating with greater and greater rapidity as time passes.

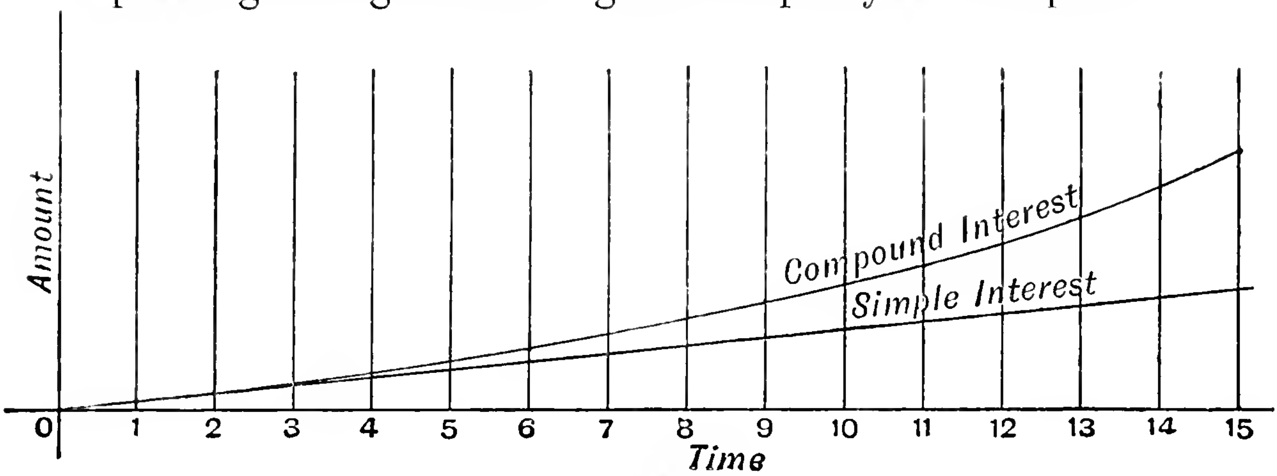


FIG. 47.

One is an A.P., the other a G.P.; one is a straight line law, the other a compound interest law. One proceeds by constant increment, the other by constant factor. (See pages 404, etc.)

As an example consider interest at the rate of ten per cent. per annum, and tabulate the value of £1000 after successive years on the two systems.

Time.	Value at 10 % Interest.		Factor.
	Simple.	Compound.	
at start	£1000	£1000	1
1 year	1100	£1100	1·1
2 years	1200	£1210	$(1·1)^2 = 1 + ·2 + ·01 = 1·21$
3 years	1300	£1331	$(1·1)^3 = 1·331$
4 years	1400	£1464. 2s.	$(1·1)^4 = 1·4641$
5 years	1500	£1610. 10s. 2½d.	$(1·1)^5 = 1 + ·5 + ·10 + ·01 + ·0005$
6 years	1600		$(1·1)^6$ [+ ·00001 = 1·61051
10 years	2000	£2593. 15s.	$(1·1)^{10}$
15 years	2500		
20 years	3000		
50 years	6000		
100 years	11,000	£13,781,000	$(1·1)^{100}$
200 years	21,000	£189,910,000,000	$(1·1)^{200}$

The *rate* of interest plotted in fig. 47 is arbitrary, and depends upon the original sum or 'principal'; this would appear below the base line, and the diagram represents only its growth. The curve on page 179 is really the same curve, but to get the right aspect it must be looked at through the paper; and the scale of plotting is unsuitable.

The right hand column of the above tabulated numbers contains the factors by which the original sum must be multiplied to give the amount at compound interest. It is hoped that the binomial coefficients will be recognised.

It may be noticed that during the length of an active lifetime the difference between simple and compound interest is not extravagant, even at so high a rate as ten per cent; but that if a sum is locked up during a long minority, or if otherwise the interest be left to accumulate for a long time without being contemporaneously expended, the growth by compound interest becomes enormous.

The operation of a "sinking fund" for the annihilation of great debts can thus be illustrated. But it is not to be supposed that interest accumulates, without effort, automatically. It is the result of human skill, brains, labour, and management.

PART II.

MISCELLANEOUS APPLICATIONS AND INTRODUCTIONS.

CHAPTER XXXVIII.

Illustrations of important principles by means of expansion by heat.

EVERYONE has seen a telegraph wire by the side of a railway and observed the peculiar effect of its sag, as the train passes along, when such a wire is near the window; it seems to be moving up and down. A wire or rope or chain stretched between two posts cannot be perfectly straight, but sags, something like the top of a lawn tennis net.

In hot weather the sag of a given span of wire is greater, in cold weather it is less; because the material expands by heat and contracts by cold. Suppose the length of a wire on a certain span is l during a night of light frost, then by noon, when the sun has been up some time, it will have increased to l': the increase of length being $l' - l$.

This notation for the same kind of quantity under different circumstances, by means of the same letter with a dash affixed to it, is in constant use, and must be grown accustomed to; the new length should be read "l-dash"; the increase should be treated as a single quantity and should be read "l-dash minus l." And here it is desirable to remark that all mathematics is intended to be read, and that it is good and necessary practice to read it.

Algebra is a language—a very expressive language; and although it appeals primarily to the eye, it should be made to appeal to both eye and ear, that is it should always be "read," if only to oneself. It is a great mistake to treat it as a silent language and only to look at it, beginners must learn to read it; and it would be well now to turn back and read aloud all the equations and other algebraical expressions we have employed so far. It cannot be done properly without a good deal of practice. Everything written on a blackboard by a teacher should be spoken also.

Now consider what the increase in length depends on. In the first place it depends on the original length of the wire. No one would expect a wire a few inches long to elongate as much as one a few hundred yards long. It is only common sense to realise that every yard of the wire will elongate an equal amount under the same conditions, and that therefore 20 yards will lengthen 20 times as much as one yard; so $l' - l$ is proportional to l. Next it will depend on the change of temperature, which we may treat as a single quantity and call $T' - T$, where T was the original temperature and T' the new temperature.

We have no guarantee that the lengthening is *proportional* to the rise of temperature, but it is a natural assumption to begin with, and will have to be corrected if necessary later. We can assume that it expands twice as much for two degrees' rise as it does for one degree, and ten times as much for ten degrees' rise. There are only a few substances for which this is really and precisely true, but, for all, it is a rough approximation to the truth, and will do for the present.

Lastly, the lengthening will depend on the material of which the wire is composed. If it were a copper wire it would be found to expand more than if it were of iron. Every material has its own "expansibility," by which is meant the rate of expansion, or increase per unit length per degree rise of temperature.

If we denoted the "expansibility" of the material by k, we should be able to express in one line all we have so far said, thus:

$$l' - l = kl(T' - T), \qquad (1)$$

for this asserts that the lengthening is dependent on and proportional to three factors, viz. (i) a constant representing the properties of the material, (ii) the length of the piece of that material which is under consideration, and (iii) the rise of temperature or the warming to which it has been subjected.

Of the three factors, k may be styled a "constant" to be determined by experiment in the laboratory, a thing depending on the properties of a material, and beyond our control, except in so far as we can select the material; l is an arbitrary constant entirely in our control, depending only on what we choose to attend to. We might observe the lengthening of the whole of a span of wire, or of any portion of it, or we might select spans of different length, or we might cut off a bit and attend only to that. $T' - T$ represents the change or variation of a variable quantity, in this case temperature; it is sometimes called the independent variable, for its changes go on independently of any of the other things we have considered, and the change of length is dependent on it. One could hardly make the converse statement and say that change of temperature was caused by change of length, even though the length was that of a thermometer column (though it might be rash to stigmatise even this statement as absurd under all circumstances: there are things which get warm when and by reason of being stretched), but it is extremely natural to say that the change of length is caused by change of temperature; so the cause is called the independent, and the effect the dependent, variable. By some people the names "principal" and "subordinate" variables are preferred to "independent" and "dependent."

A change of temperature might be caused out of doors by the appearance and disappearance of the sun, or by a change of wind; in a laboratory it could be caused by the application and removal of flame, or of an electric current, or of some other means of heating. Anyway it is to be observed by a thermometer of some kind, and $T' - T$ may be considered as being measured by the rise of the thermometer.

$l' - l$ is the change of the dependent or subordinate variable; and its dependence might be conveniently indicated by putting the two variables on one side of the above equation, and the constants on the other, as for instance:

$$\frac{l' - l}{T' - T} = kl. \qquad (2)$$

To emphasise the fact that the two terms $T' - T$ represent a thing which is really one quantity viz. a warming, a difference of temperature, it is convenient to have a single symbol for it; and the symbol usually chosen is an abbreviation for difference of temperature, namely δT or dT; meaning

$$\text{diff. of temp.} = T' - T,$$

$$\text{or diff. } T = T' - T, \text{ or simply } dT = T' - T.$$

This mode of expression is very handy and extraordinarily convenient. It can be applied of course to all kinds of quantities, so $l' - l$ may be written "diff. length," or dl; wherefore our two above numbered forms of a proportionality statement become abbreviated into

$$dl = kl\,dT \qquad (1)$$

and

$$\frac{dl}{dT} = kl \qquad (2)$$

respectively.

These equations are labelled like the previous ones because they say precisely the same thing as the others did, and in the same way. It is generally understood that the symbol d

is used for a difference only when it is an infinitesimal difference. For finite differences *delta* l is used, or simply $l' - l$. Form (1) gives explicitly the change of length in terms of the original length and the change of temperature; form (2) gives the ratio of the changes of the two variables in terms of the constants—viz. the expansion-property of the material, and the length selected for observation.

Another way of writing the equation is often useful, in which the expansion per unit length is explicitly attended to: that is the lengthening by heat of any one yard or foot or metre of the wire, without regarding the whole wire. To get this we have only to divide the length out, and so get

$$\frac{dl}{l} = kdT, \quad\text{..........................}(3)$$

a quantity which is often technically referred to as "the expansion"; it is *defined* as the ratio dl/l, and it is equal to the expansibility multiplied by the rise of temperature.

Let no beginner suppose that these various forms of the equation are different. They are all essentially the same, but they emphasise features differently; just as in any language a sentence may be recast so as to say the same thing with various emphases. Never forget to regard algebra as a language, in which statements of singular definiteness and precision can be compactly made.

Now suppose the temperature fell instead of rose: the expression $T' - T$ would be negative, and we might sometimes choose to denote the fall of the temperature by $T - T'$. At the same time most substances contract with cold, and so $l' - l$ would also be negative, and the contraction could be written $l - l'$ or $-dl$; but usually dl and dT would not have the negative sign actually prefixed to them, it would be sufficient to say that dT and usually dl are both negative, for the case of a fall of temperature.

Now let us begin again, and look at the matter afresh and in a still simpler manner. Take a rod of length 1, that is to say 1 foot or one metre or one inch in length, at a temperature T, and warm it one degree. Its length will now be increased by an amount we will call k, meaning k feet or metres or inches, according to our choice of unit length.

Warm it 2 degrees and its increase of length will be $2k$, and so on, as shown thus:

its length being	1	at temperature	T,
its length is	$1+k$	,, ,,	$T+1$
,, ,,	$1+2k$	,, ,,	$T+2$,
,, ,,	$1+3k$	,, ,,	$T+3$,
	......		......
,, ,,	$1+nk$	,, ,,	$T+n$.

This temperature $T+n$ we will call T' so that $n = T'-T$.

If the original rod had been originally of length l instead of length 1 and had been all of it treated alike, every unit would have expanded by the same amount, so the final length would in that case be $l(1+nk)$, which we may call l'.

Hence $$l' = l(1+nk)$$

or $$l'-l = lnk$$

$$= kl(T'-T),$$

thus arriving at the same result as before, which we will now write in any one of four equivalent ways, *e.g.*

$$dl = kldT, \quad\text{.........................}(1)$$

$$\frac{dl}{dT} = kl, \quad\text{.........................}(2)$$

$$\frac{dl}{l} = kdT, \quad\text{.........................}(3)$$

$$\frac{1}{l}\cdot\frac{dl}{dT} = k. \quad\text{.........................}(4)$$

The last may be taken as a definition of the expansibility k, and shows the principle of what we must do in the laboratory in order to measure it.

We must take a rod or wire or something convenient of the given material, and measure either its whole length l, or a length l between two marks or scratches on it; then we must subject it to a *measured* rise of temperature, and observe the increase in length of the chosen portion carefully, with a microscope or micrometer by preference.

Then it is best to cool it down again and see that the rod recovers its original length; and then the warming can be repeated, and the increase in length observed again, and so on several times, to avoid accidental errors and to get the true reading as nearly as we can.

Thus the three required quantities dl, dT, and l, are all measured; and, dividing dl by l and by dT, we get the expansibility of the material as a result.

These are not laboratory instructions, and accordingly little or nothing shall here be said about the practical mode of overcoming difficulties. Suffice it to say that the readiest mode of securing measurable differences of temperature, is by making use of properties of substances designated by such phrases as boiling oil, molten lead, melting ice, boiling water, condensing steam, and the like; and that the chief precautions needed, in order to measure with precision the expanded length, are those which shall guard the measuring scale, or standard of length, from being likewise affected by the high temperature of the rod to which it has in some sort to be applied.

With this hint the somewhat elaborate arrangements depicted in text books of Physics can be appreciated.

This matter has been gone into at some length, because it is typical: it is always worth while to master a type, and nothing is gained by haste.

Examples.

A bar of iron 10 yards long expands ·444 inch when taken out of ice and put into steam or boiling water; what is its expansibility? *i.e.* what is the increase per unit length per degree for iron; meaning by a degree Centigrade the hundredth part of the interval between freezing and boiling water, and by a degree Fahrenheit the 180th part of that same interval.

Answer. $\frac{\cdot 444}{360 \times 100} = \cdot 0000123$ per degree Centigrade,

or ·00000683 per degree Fahrenheit.

The numerical result is worth remembering as specified in the Centigrade scale of thermometry, which is the most used for scientific purposes. Observe that there are 4 ciphers before the significant figures, which happen to be the first three natural numbers, and so are quite easy to remember; the amount is about $1\frac{1}{4}$ in a hundred thousand, or about 12 parts in a million: meaning that iron expands this fraction of its length for each Centigrade degree rise.

Brass would give a number about 18 instead of 12; and zinc, which is one of the most expansible metals, would expand nearly 25 parts, or just double as much as iron. Platinum however, and glass, would have been found to expand only about 8 or 9 parts in a million, per unit length per degree Centigrade.

If a material expanded 1 per cent. of its length for a rise of 100°, its **expansibility** would be ·0001. If it expanded $\frac{1}{2}$ per cent. for a rise of 250°, its expansibility would be ·00002, which lies between that of brass and that of zinc.

Cubical Expansion.

It would be a mistake to suppose that a rod increases in only length when heated: it swells in every direction, just as if it were slightly magnified. Its increase in length is most noticeable, because that was originally its greatest dimension, but its increase in thickness is proportionately as much. Thus if a bar were a yard long and an inch thick it would expand in length 36 times its increase in thickness, but its proportional expansion, its $\frac{dl}{l}$, would be the same in every direction.

Consider an iron plate 10 metres long, 1 metre broad, and 1 millimetre thick, and let it be warmed 406 degrees. Its linear expansion is

$$406 \times \cdot0000123 = \cdot00500,$$

or five parts in a thousand (or the half of 1 per cent.).

So its increase in length is 5 centimetres;

its increase in breadth is 5 millimetres;

its increase in thickness is ·005 millimetre,

or 5 millionths of a metre,

or 5 mikrons,

a mikron (sometimes spelt micron) being a convenient unit for microscopic work, and being sometimes *in*conveniently denoted in biological books by the symbol μ. *Units or standards* should be expressed in words; symbols are never used for them by mathematicians.

What is the increase in area and in bulk of such a plate when so heated?

The first thing to learn is that we must *not* take the increases and multiply them together. (Cf. p. 293.)

The increase in area is *not* 5 centimetres × 5 millimetres.

The increase in volume is *not* 5 centimetres × 5 millimetres × 5 mikrons.

But the new area is 1005 × 100·5 sq. centimetres, whence the increase in area is 1002·5 sq. centimetres.

The new volume is $1005 \times 100 \cdot 5 \times \cdot 1005$ cubic centimetres

$$= 10^3 \times 10^2 \times 10^{-1} \times (1 \cdot 005)^3$$
$$= 10^4 \times 1 \cdot 015075$$
$$= 10150 \cdot 75 \text{ c.c.}$$

The old volume was $10^3 \times 10^2 \times 10^{-1} = 10^4$ c.c., so the increase in volume is $150 \cdot 75$ c.c.

But, as usual, there is a quicker and better mode of making the numerical calculation, by first treating it algebraically.

Let $l' = l(1+kt)$ be the new length,

$b' = b(1+kt)$ the new breadth,

and $z' = z(1+kt)$ the new thickness,

where t stands for the rise of temperature $T' - T$.

Then the new volume is

$$l'b'z' = lbz(1+kt)^3;$$

that is, calling the old volume V and the new volume V',

$$V' = V(1+kt)^3$$
$$= V(1 + 3kt + 3k^2t^2 + k^3t^3);$$

but, since kt is a small quantity, this is, to a first approximation,

$$V' = V(1+3kt),$$

or $V' - V = 3kVt = 3kV(T' - T)$,

or $\dfrac{dV}{dT} = 3kV$, or $\dfrac{dV}{V} = 3kt$;

a result which can be expressed by saying that the cubical expansibility, viz. $3k$, is three times the linear expansibility.

Similarly the coefficient of superficial expansion is twice the linear.

This is equivalent to neglecting squares and cubes of small quantities; and for most purposes that can safely be done in cases of solid expansion. Hence to do the above sum, very approximately, all that is necessary, after observing that the linear proportional increase is ·005, is to say:—The original area = 10 square metres, so the increase of area is $2 \times \cdot 005 \times 10$ square metres

$$= 0{\cdot}1 \text{ sq. metre} = 1000 \text{ sq. centimetres approximately.}$$

The original volume is

$$10 \text{ metres} \times 1 \text{ metre} \times \cdot 001 \text{ metre} = \cdot 01 \text{ cubic metre}$$
$$= 10{,}000 \text{ c.c.}$$

so the increase of volume is $3 \times \cdot 005 \times 10{,}000$ c.c.

$$= 150 \text{ c.c. approximately}$$

CHAPTER XXXIX.

Further Illustrations of Proportionality or Variation.

One of the most important things to understand, in order to be able to apply elementary mathematics to simple engineering facts, is the law of simple proportion. Two quantities are said to be proportional, or to **vary** as each other, if they are both doubled when one is doubled and if they vanish together.

Thus for instance the stretch of an elastic and the force of its pull are proportional. For first of all they vanish together: if the elastic is not pulled it is not stretched. Secondly, if it has been stretched with a certain pull, and you double the pull, the stretch also will be found to be doubled. You can try this by hanging up an elastic or a spiral spring and loading it with different weights. As the weight increases, the stretch increases, and in a simply-proportional manner. This is the principle of a spring balance.

Take all the load off, and the pointer returns to zero, indicating no stretch. Observe, it is not the *length* of the elastic that is proportional, for that does not become zero, nor is it doubled when the load is doubled, but it is the *stretch* or increase of length that is proportional to the load. Suppose however there had been some irremovable load on all the time, as indeed there is, for the spring or elastic itself has a trifle of weight of its own, how do we allow for that? Answer: by always attending to the variable part of the load only, just as we attend to the variable part of the length only. The "load" must really signify the load *added* or subtracted; and the stretch corresponding thereto signifies the increase, or it may be the decrease, of length which accompanies that variation of load; so that instead of saying the length l varies as w the load, which is not true, we ought to say, change of length varies as change of load, or dl varies as dw, which is quite true; unless indeed the spring is overloaded and permanently strained or injured so that it cannot recover; or, in other words, unless it is not perfectly elastic. So long as it is perfectly elastic, the law of simple proportion holds; and the test of whether it is perfectly elastic or not is to see if it can completely recover when the load is removed.

Some substances stand a great deal of loading, such as steel; some stand only a little without giving way, like glass or copper; and some stand hardly any, or none, like lead or straw or dough.

There are two methods of giving way, one by breaking, like glass, the other by permanently bending, like lead. There is plenty to be learnt about all these things, but the time for learning them will come later. All that we have to note at present is that the law of proportion is not to be expected to be verified when the substance experiences a permanent set, or deformation of any kind, from which it cannot recover; and of course not when it is broken.

The law holds "within the limits of elasticity," and it is known as "Hooke's law," because that great and ingenious man Robert Hooke experimented on it and emphasised its simplicity and convenience more than two centuries ago.

You may think that it is so simple as not to be much of a law of nature; but you will find that all the most fundamental laws are simple. Simplicity and importance may quite well go together, though there is perhaps no necessary connexion between them.

Now take another example of simple proportion. Let a balloon ascend at a perfectly steady pace from the ground. Its height is proportional to the time which has elapsed since it started. In one second it went up let us say a yard. In a minute it will go up 60 yards, and in an hour 3600 yards, provided the same upward speed was all the time maintained. If it went sometimes faster and sometimes slower, the simple proportionality would not hold. The height and the time vanish together, for we began to reckon time at the instant it was let go, and we were careful to measure height always to that same point of the balloon which touched the earth at the moment of letting go.

But now take an example where simple proportion does not hold between two connected variable quantities.

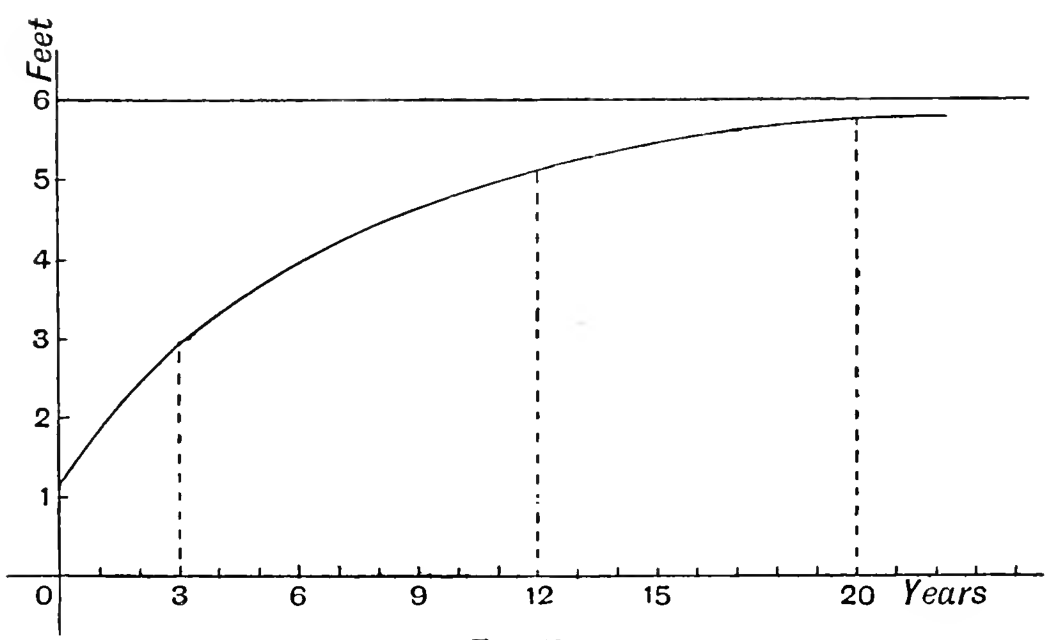


FIG. 48.

The height of a child depends upon his age, and increases with his age, but it is not proportional to it: in other words, his height does not *vary with* his age, in the technical sense.

For first of all they do not vanish together: the child had some length when he was born; and next, they are not doubled together. A child has not twice the height at 6 that he had at 3.

If we were to plot age and height together, it would be instructive, and the result might be something like fig. 48, where age is plotted horizontally and height vertically.

The point *O* is called the origin, and represents the epoch of birth. If the curve passed through this point *O*, zero initial length would be signified; so the curve does not pass through it, but starts above it at the infant's length at birth: say fourteen inches.

Such a curve is not simple proportion at all. It is easy enough to understand, but the law represented by the curve is not a simple one.

Simple proportion would be represented, on the same plan, by a straight line through the origin; as for instance if the stretch and the load of an elastic thread or spiral spring were plotted: they vanish together. (Fig. 49.)

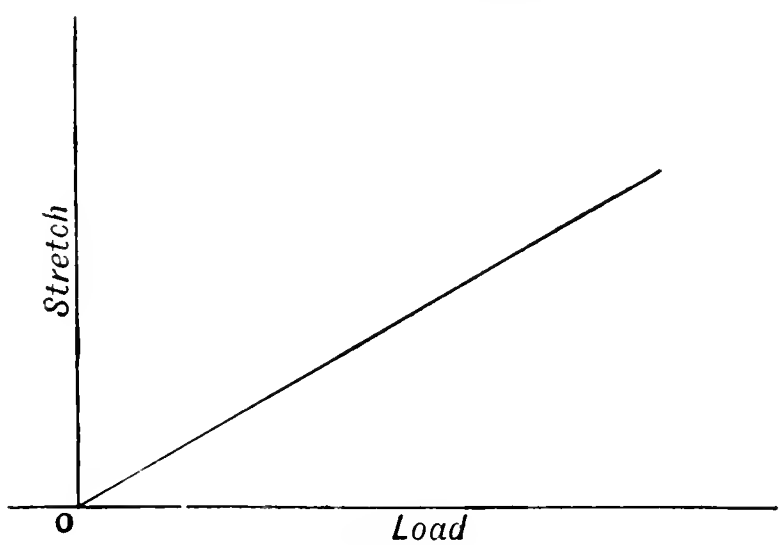


FIG. 49.

But suppose the two quantities did not vanish together, we might still have them plotted as a straight line. For instance, the length of a rod at different temperatures (or the total length of a piece of elastic under different loads):

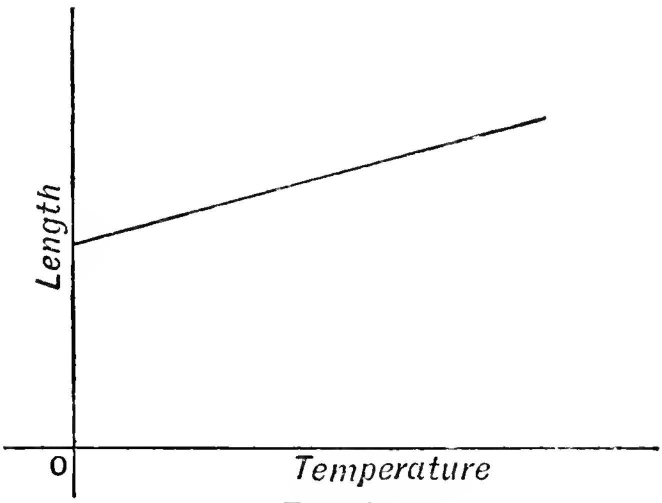


FIG. 50.

and this, though it is not exactly simple proportion, is the next thing to it, and is sometimes called "a straight line law." (Fig. 50.)

It can be made simple proportion by deducting a constant, by deducting the original length for instance, $l' - l = klt$, whereas if the length l had not been deducted it would have been expressed by

$$l' = l + lkt,$$

and this is characteristic of a straight line law.

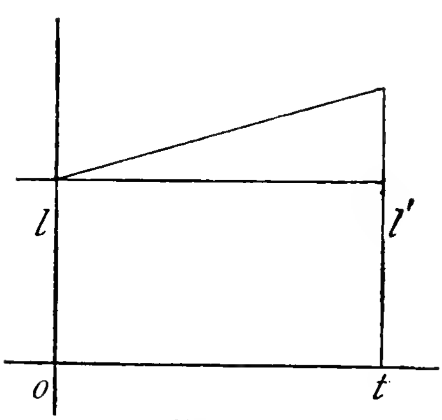


FIG. 51.

In general a straight line law is represented by

$$y = a + bx,$$

whereas simple proportion would be

$$y = bx$$

without the constant a.

So by subtracting the constant a a straight line law can always be expressed as simple proportion. So it can if we attend to *changes* only. For let y become y', and x become x', while a and b remain constant all the time, we should have

$$\begin{aligned} y &= a + bx, \\ y' &= a + bx', \\ y' - y &= b(x' - x), \\ \text{or } dy &= b\,dx, \\ \text{or } \frac{dy}{dx} &= b, \end{aligned}$$

and the a has disappeared. The two differences, dy and dx, are simply proportional; for they increase together in a constant ratio, and they vanish together: one cannot become zero without the other.

The weight of a boy is not represented by a straight line, even if his weight at birth is deducted. His law of growth is not a straight line law but a more complicated law: it could be plotted as a curve, from successive observations.

Any law can be expressed by a curve; thus we might have a parabolic law, meaning that the curve of plotting is a parabola as nearly as can be told.

A parabolic law is expressible in algebra thus,

$$y = a + bx + cx^2,$$

and it would be instructive for children to plot the curve represented by this equation, and see what it looks like. To carry out the plotting we must be told the values which are to be attributed to a, b, and c, and can arrange the scale to suit.

Thus let $a = 4$, $b = 1$, and $c = \frac{1}{4}$.

Make a table of corresponding successive values of x and y.

When $x = 0$,	$y = a$,	or in this case 4;
$x = 1$,	$y = a + b + c$,	or in this case $5\frac{1}{4}$;
$x = 2$,	$y = a + 2b + 4c$,	or 7;
$x = 3$,	$y = a + 3b + 9c$,	or $9\frac{1}{4}$;
$x = 4$,	$y = a + 4b + 16c$,	or 12;
$x = 5$,	$y = a + 5b + 25c$,	or $15\frac{1}{4}$;
$x = 6$,	$y = a + 6b + 36c$,	or 19; etc.

Plot this, and it looks somewhat thus:

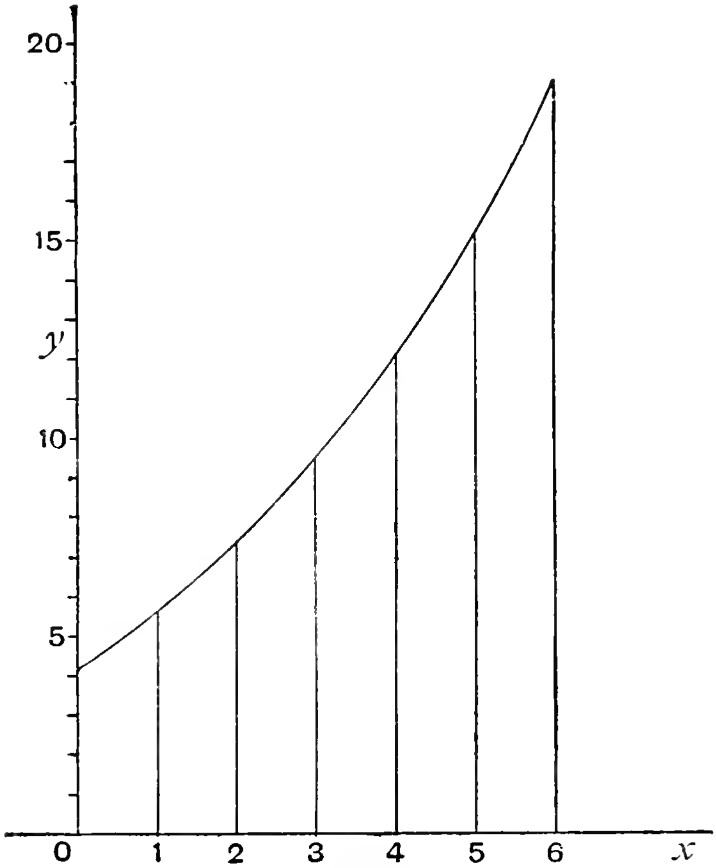


FIG. 52.

But we need not necessarily limit ourselves to the positive side of the vertical axis; we might ascertain and plot values of y when x is negative, otherwise the curve is incomplete. You could hardly tell it was a parabola from its appearance so far.

When $x = -1$,	$y = a - b + c$	$=$	$3\frac{1}{4}$;
$x = -2$,	$y = a - 2b + 4c$	$=$	3;
$x = -3$,	$y = a - 3b + 9c$	$=$	$3\frac{1}{4}$;
$x = -4$,	$y = a - 4b + 16c$	$=$	4;
$x = -5$,	$y = a - 5b + 25c$	$=$	$5\frac{1}{4}$;
$x = -6$,	$y = a - 6b + 36c$	$=$	7;
$x = -7$,	$y =$		$9\frac{1}{4}$;
$x = -8$,	$y =$		12;
$x = -9$,	$y =$		$15\frac{1}{4}$;
$x = -10$,	$y =$		19; etc.,

every value on this side being equal to a certain value for another x on the other side. The whole curve is symmetrical about a vertical axis through $x = -2$.

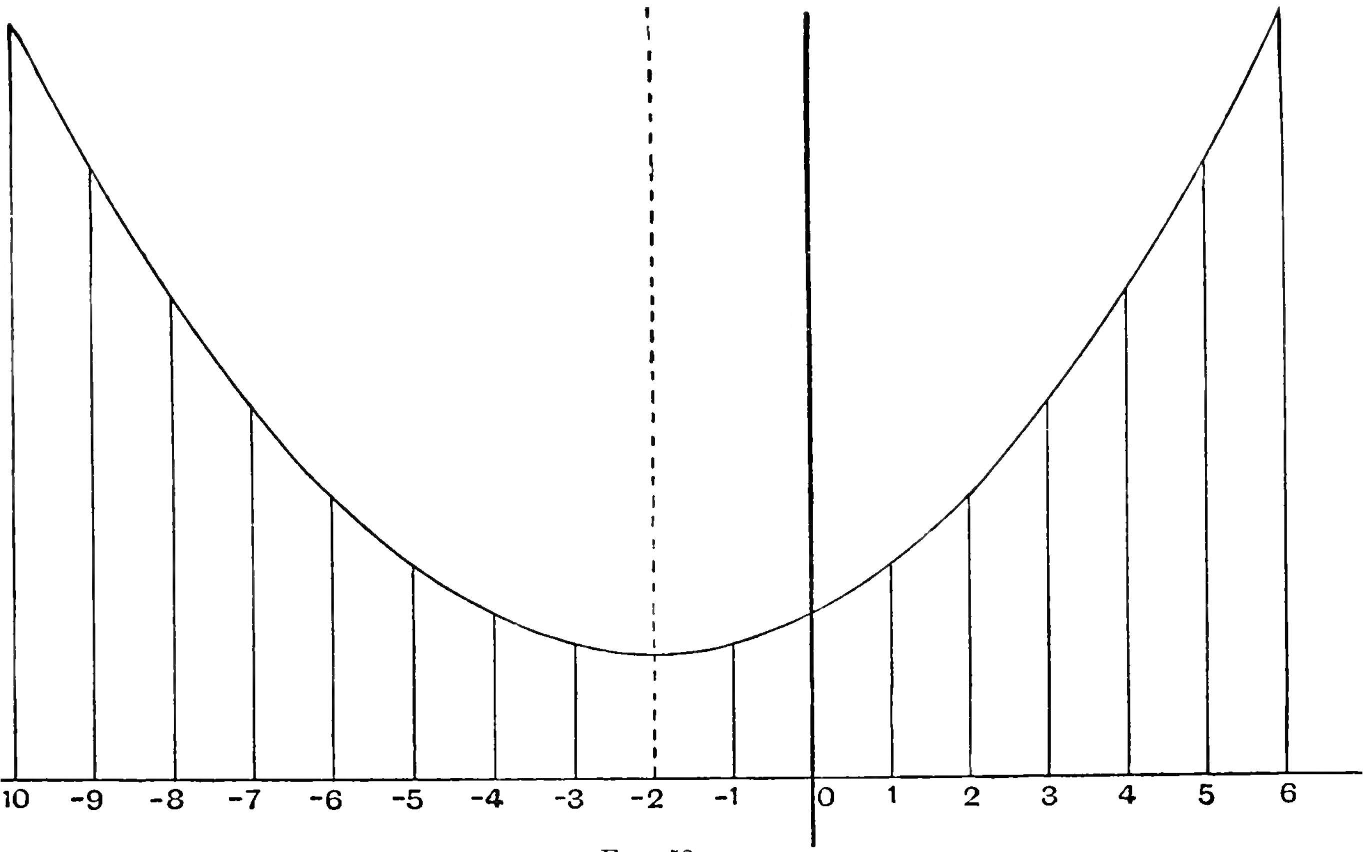


FIG. 53.

The dotted line is called the axis of the parabola. (Fig. 53.)

Another example of what may be a straight-line-law is the slope or gradient of a railway. At a certain place it may be said to rise 1 in 30, for instance; meaning that if you go 30 feet along the railway you have ascended 1 vertical foot; if you go 30 yards you have ascended 1 vertical yard, and so on.

A uniform gradient is naturally plotted by a straight line, and if the vertical height is called y while the horizontal distance is called x, the gradient is approximately denoted by $\frac{dy}{dx}$. Not exactly, because the denominator is usually measured along the sloping railway, and not horizontally.

Either way of measuring the gradient is a good method; and sometimes one is used, sometimes the other. If the slant distance is called ds, the two chief ways of measuring slope are $\frac{dy}{dx}$ and $\frac{dy}{ds}$, respectively.

Except when the slope is steep, the difference between these two methods of measuring it is not marked:—the connexion between the two methods is easily shown by a diagram.

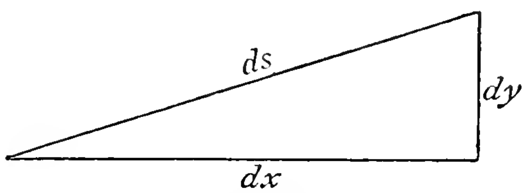


FIG. 54.

A considerable but feasible slope for an ordinary railway would be a gradient of 1 in 30, that is to say $\frac{dy}{ds} = \frac{1}{30}$.

Often it is not more than 1 in 100. The actual gradient is frequently written up on low posts by the side of the line.

But take the case, impossible for a practical railway, where the slope is 45°, which would be a steep mountain side, dy and dx are in that case equal, and $ds = \sqrt{2}$ either of them,

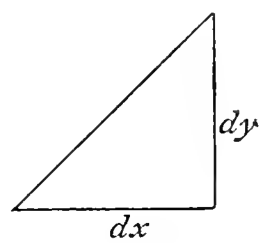


FIG. 55.

so $\frac{dy}{dx} = 1$, whereas $\frac{dy}{ds} = \frac{1}{\sqrt{2}}$.

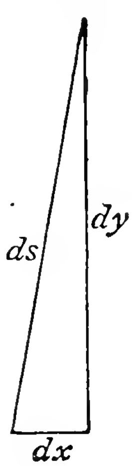


FIG. 56.

Take the case of a precipice or a steeple, almost vertical, so that dx is extremely small.

dy and ds are now nearly equal and dx is nearly 0; in the limit quite zero.

So approximately, for an angle nearly 90°,

$$\frac{dy}{ds} = 1 \text{ and } \frac{dy}{dx} = \frac{1}{0} = \infty,$$

or at any rate a very great number; in the limit quite infinite.

Take the case of that famous right-angled triangle with commensurable sides, and express the slope of its hypothenuse:

$$\frac{dy}{dx} = \frac{3}{4}, \qquad \frac{dy}{ds} = \frac{3}{5},$$

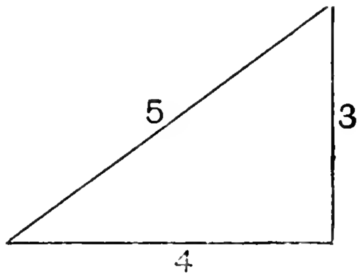


FIG. 57.

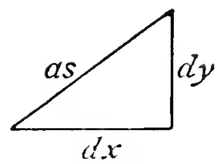


FIG. 58.

and of course always $ds^2 = dx^2 + dy^2$,

so that
$$\frac{dy}{ds} = \frac{dy}{\sqrt{(dx^2+dy^2)}} = \frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$
$$= \frac{dy}{dx}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{-\frac{1}{2}}.$$

It is often convenient to measure slope in yet another way, viz. by the angle of slope, which we denote by α or θ; then the ratio of the height to the slant length, $\frac{dy}{ds}$, is called the **sine** of the angle, and is written sine θ or $\sin\theta$, while the ratio of the height to the base, $\frac{dy}{dx}$, is called the **tangent** of θ, and is written $\tan\theta$. So we see from the above that an angle whose sine is $\frac{3}{5}$, or $\cdot\dot{6}$, has a tangent whose value is $\frac{3}{4}$ or $\cdot 75$: also that the sine of 45° is $\frac{1}{\sqrt{2}}$, while the tangent of 45° is 1, and that $\sin 90° = 1$, while $\tan 90° = \infty$. (Cf. fig. 56.)

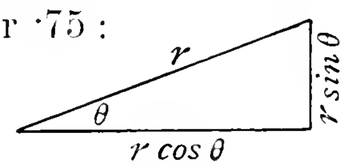


FIG. 62.

On the other hand, when the slope is very slight, the sine and tangent are about equal, and it does not much matter which measure of the slope we employ. (Cf. fig. 54.)

Ultimately when the slope vanishes they also vanish, and vanish on terms of equality, so that sin 0 = tan 0 = 0.

Here is a case where two things vanish together but are by no means proportional; they are approximately proportional, or indeed equal, for small angles, but the tangent increases faster than the sine; and as the angle grows, it increases very much faster; so that, by the time the sine has reached unity, the tangent has gone, with a rush, to infinity.

Plotting them they would look thus: where one curve represents the sines, and the other the tangents, of angles from 0° to 90°

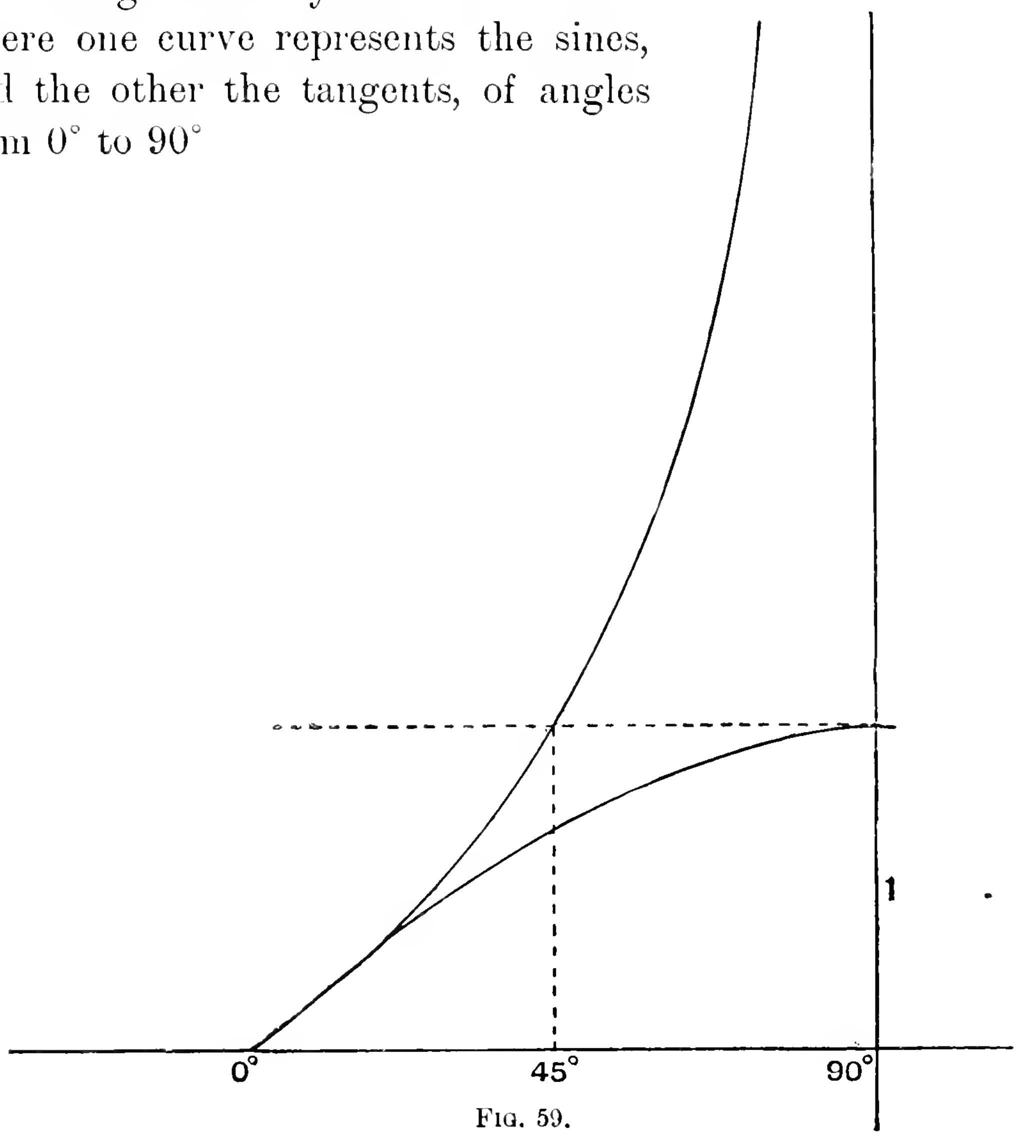


FIG. 59.

It may be worth showing, even at this stage, what suggested these curious names. The name "tangent" especially sounds curious when applied to a ratio. The idea arose from drawing a circle round an angle and seeing all the different ways in which it might be measured.

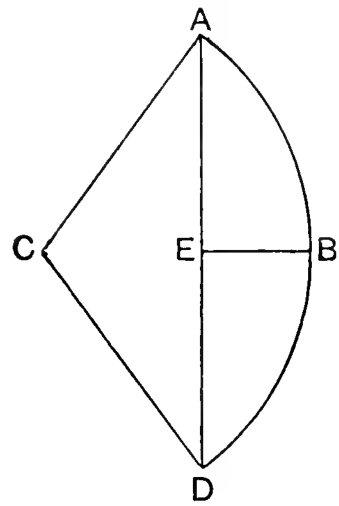


Fig. 60.

In this figure the angle is at C, and AD is a bit of a circle with centre C. This figure long ago suggested a bow and arrow, hence EB was called the sagitta, and ABD was called the arc: the string AED is called the chord, and half of it under certain restrictions is called the sine, presumably because the point E of the string is pulled to the breast before releasing the bow. The tangent can then be measured as part of a line drawn through B, touching the circle, when the circle is drawn of unit radius.

Let CA or CB equal 1 on any arbitrary scale; then C or ACB is the angle, α,

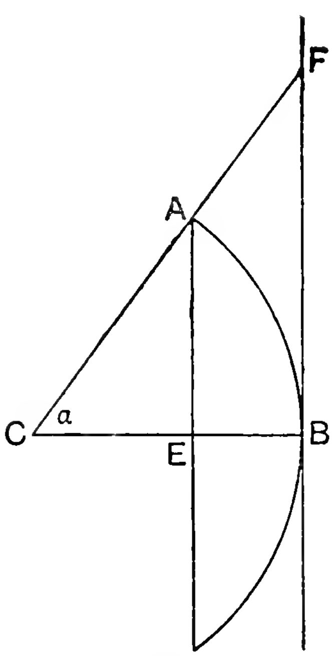


Fig. 61.

AE is its sine,
AB is its arc,
EB is its sagitta,
and BF is its tangent,

always provided CA or CB are equal to 1, that is taking the radius of the circle as unity.

The size of the circle is quite arbitrary: any length whatever may be chosen as a unit of measurement.

But it is desirable to bear in mind that angles are not measured by length but by ratio; and accordingly the statement that

$$\text{angle} = \frac{\text{arc}}{\text{radius}} = \frac{AB}{AC},$$

or intercepted part of arc ÷ radius, is a better statement than to say that

$$\text{angle} = \text{arc when radius} = 1.$$

And to say that

$$\sin \text{angle} = \frac{\text{intercepted perpendicular}}{\text{radius}} = \frac{AE}{AC},$$

is better than saying that it equals AE when AC equals 1.

So also

$$\text{tangent of } C = \frac{\text{intercepted portion of tangent}}{\text{radius}} = \frac{FB}{BC}.$$

It may be worth while also to state here that the length of that boundary line of the angle which cuts the circle and is produced to meet the tangent, viz. CF, is called the "secant" when the circle is of unit radius; or in general, dividing by the radius,

$$\text{secant of angle } C, \text{ or sec } C, = \frac{CF}{CB}.$$

These different fractions or ratios are all *measures* of the angle: they are quite independent of the size of the circle or of any linear dimension whatever. They indicate angular magnitude alone.

In any right-angled triangle, if the length of the hypothenuse is called r, and the angle at the base be called θ, then the length of the base is, by definition of cosine, equal to $r \cos \theta$; that is to say, it is the length r multiplied by a proper fraction, which fraction is called the cosine of the angle θ.

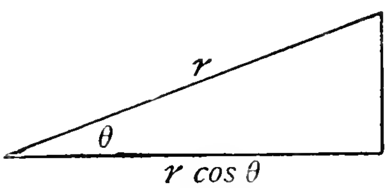


FIG. 63.

The base may be thought of as the **projection** of r on to a direction inclined at the angle θ to it; the *shadow* as it were of a slant rod of length r thrown by vertical rays of light on a horizontal ground (fig. 64).

If the projection were made in a direction at right angles to the first, then the projection is $r \sin \theta$, provided θ still means the same angle as before. So that the sides in any right angled

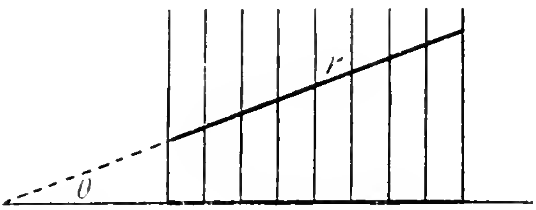


FIG. 64.

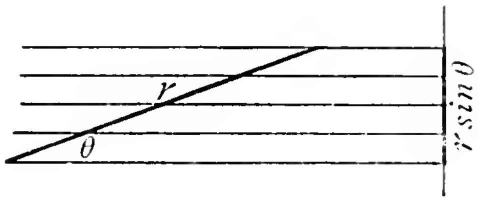


FIG. 65.

triangle are related as in the diagram (fig. 62); and it is obvious, by definition of the tangent ratio, that

$$\frac{\sin \theta}{\cos \theta} \equiv \tan \theta.$$

It also follows, by Euc. I. 47, that

$$(r \sin \theta)^2 + (r \cos \theta)^2 = r^2,$$

or that
$$(\sin \theta)^2 + (\cos \theta)^2 \equiv 1,$$

a fundamental identity.

One peculiarity of angular magnitude is that it is unmagnifiable. Look at an angle through a magnifying glass, and though its sides lengthen, the angle continues constant. A right-angle, for instance the corner of a book, continues a right-angle, and 45° continues 45°. A degree is always the 360th part of a circle, however big the circle; a quarter of a revolution, or a right angle, is always 90°; and so on. The size of the divisions of a circle change when it is magnified, but their number remains constant.

Number is another thing which is unmagnifiable. Magnify 3 oranges, the oranges look bigger, but they still look 3.

So when a plate expands with heat, if it is uniform, any angles it may have remain constant. If there were a hole in the plate, the hole would expand just as if it had been filled with solid; its boundary line might have been drawn with ink on a similar solid plate. Everything expands with heat as if it were looked at through a weak magnifying glass. So a hollow space is not encroached upon by expanding walls, but is enlarged as if it were full of substance. A hollow bulb, for instance, has a greater capacity when heated than it had before. It may not necessarily hold more fluid, not more weight or mass of fluid, for the fluid might expand still faster than the solid, but it holds a greater volume. A thermometer bulb containing mercury is in this predicament; and what we observe, when the thermometer rises, is the apparent expansion of the whole bulb-full of mercury swelling in the only direction open to it, viz. in the narrow stem. It is called "apparent" expansion because it represents what is visible, viz. the excess of the expansion of the mercury over that of the vessel which contains it. If the vessel expanded more than the liquid, the rise in the stem, indicating the apparent expansion, would be negative; but this is hardly a possible case in practice, for all liquids expand more than any solid. Nevertheless the true or absolute expansion of a liquid is always greater than it appears to be, unless we could observe it in a vessel which did not expand with heat.

Let v be the volume of the vessel, and a the expansibility of its material, that is to say its increase in bulk per unit original bulk per degree rise of temperature, then the total swelling for any rise of temperature $T' - T$ is

$$v' - v = av(T' - T),$$

$$\text{or } dv = av\,dT,$$

$$\text{or } \frac{dv}{v} = a\,dT;$$

or, in words, the proportional expansion equals the change of temperature multiplied by the expansibility. If we assume simple proportionality between change of volume and change of temperature, it is the same thing as assuming that a is a constant; and in that case the expansion is said to vary with the temperature.

The term "varies with" or "varies as" is a technical term, and is understood to mean "is proportional to." The latter is really the better expression, for in common language two things can vary or change together without being proportional, like the age of a child and its weight, or the amount of sunshine and the cheapness of corn, or the height of a look-out man on board ship and the distance of his horizon, or the amount of oil consumed in a lamp and the brightness of the consequent light.

But the term "varies as" or "varies with" is understood to indicate more than merely changing together: it means that they vary in a simply proportional manner, so that if one is doubled or trebled or increased 1 per cent., the other is doubled or trebled or increased 1 per cent. too.

When y varies as x, in the technical sense, it can be written $y \propto x$; where $\propto$ is a mere symbol, not much used, to denote "varies as."

Or it may be written $y :: x$, and read y is proportional to x; or it may be written, the ratio $y : x$ is constant; or $y \div x$ or $\frac{y}{x}$ is constant, equal to b say; or $y = bx$.

This last is the simplest and most satisfactory mode of stating simple proportionality, b being understood to be a constant, that is something not at all dependent on the value of x and y, which are the variables.

A **straight line law** is slightly more general than this:

it includes the main idea of proportionality, but it exempts from the necessity of vanishing together;

$$y = a + bx$$

is the typical straight line law.

Either can be expressed as

$$dy = b\,dx,$$

$$\text{or } \frac{dy}{dx} = b,$$

for when we attend to variations, the constant term a has no influence, and so disappears. The constant factor b, which is only part of a term, by no means disappears; b represents the rate of increase of y with respect to x; for instance it represents the slope of the line, being the change of elevation per unit step along the base; it is in fact the 'tangent' of the slope.

The next more general law is the **parabolic law**.

$$y = a + bx + cx^2, \quad \ldots\ldots\ldots\ldots \quad (68 \text{ i.})$$

or it might be $\quad y = a - bx + cx^2, \quad \ldots\ldots\ldots\ldots \quad (66.)$

or $\quad y = a + bx - cx^2, \quad \ldots\ldots\ldots\ldots \quad (68 \text{ ii.})$

or $\quad y = a - bx - cx^2, \quad \ldots\ldots\ldots\ldots \quad (68 \text{ iii.})$

all of them parabolic, with different appearances.

The slope of such a curve is of course not constant.

To find an expression for the slope, we must take two points near together and compare the vertical with the horizontal step, that is find the dy corresponding to a given small dx. To do this we let x change to x' and y to y', and then write the relation once again for the changed values

$$y' = a + bx' + cx'^2,$$

and now subtract the old value from the new, so as to get the difference,

$$y' - y = (a - a) + b(x' - x) + c(x'^2 - x^2)$$

$$= \{b + c(x' + x)\}(x' - x);$$

$$\therefore \frac{\text{diff. } y}{\text{diff. } x} = b + c(x' + x).$$

Now if the step is made quite small, the x and x' are extremely nearly equal, so that it matters little whether we

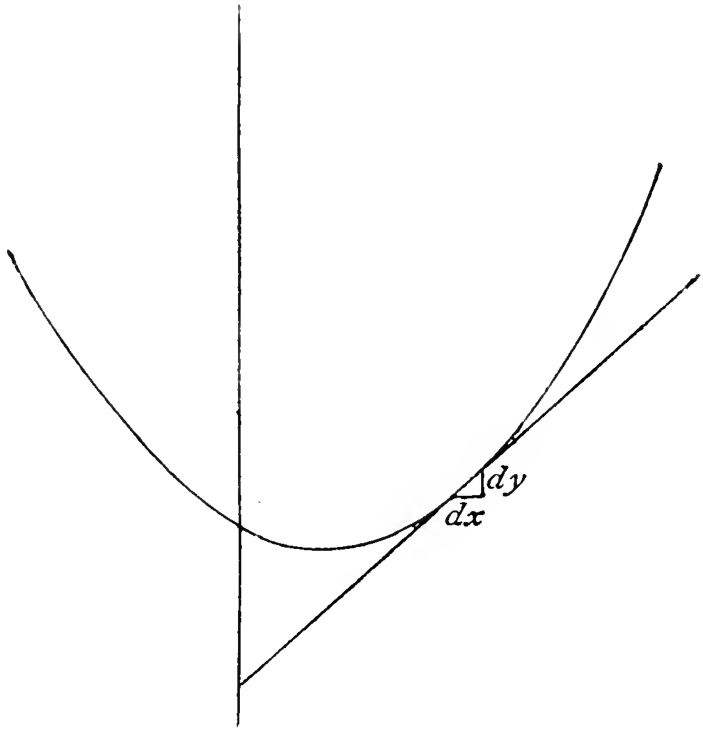


FIG. 66.

write $x'+x$ or $2x$ or $2x'$; and in the limit when the step is infinitesimal they become actually equal, and then

$$\frac{dy}{dx} = b + 2cx,$$

and this is the gradient of the curve at any point. It is not constant, but it follows a straight line law.

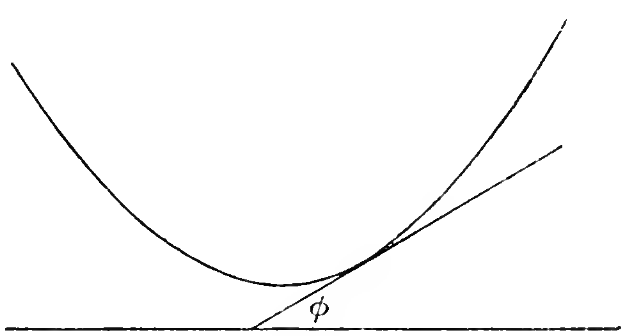


FIG. 67.

The rate of change of the angle of slope may be called the curvature. When the slope is constant, as in a straight line, the curvature is nothing, but when the slope changes, as in the last case, the curvature can be measured as the rate of change of the angle of slope per unit step along the curve. Suppose the gradient or angle of slope at any point is denoted in angular measure by ϕ, then the curvature could be defined as $\frac{d\phi}{ds}$, and that is its usual measure.

Another method of measuring it, which seems simpler but is not so satisfactory, is to indicate the slope by the tangent of the angle ϕ, that is by the gradient $\frac{dy}{dx}$, which we may denote by a single letter g, and then to denote the curvature by the rate of change of gradient per horizontal step, that is by $\frac{dg}{dx}$.

On this plan the value of the gradient for the above parabola is
$$g = b + 2cx,$$
and the rate of change of gradient is
$$\frac{dg}{dx} = 2c,$$
because if x changes to x', g changes to $g' = b + 2cx'$, and when you come to reckon the difference ratio, the b's go out:
$$\frac{g' - g}{x' - x} = \frac{2cx' - 2cx}{x' - x} = 2c.$$

It is not customary or necessary thus to introduce a new symbol g; it is neater to express $\frac{dg}{dx}$ as $\frac{d\frac{dy}{dx}}{dx}$, that is to say as $\frac{d^2y}{dx^2}$; and this it is which is equal to $2c$.

So, in the original parabolic expression
$$y = a + bx + cx^2,$$
the meaning of the constant c is half the rate of change of gradient, which is a principal term in the curvature; the meaning of the constant b is the slope, or gradient itself, especially the slope at the place where x is 0, that is to say where it cuts the axis of y; and the meaning of a is the height at which the curve cuts the axis of y, that is to say the intercept on that axis. Compare the equations on page 388 with the curves drawn.

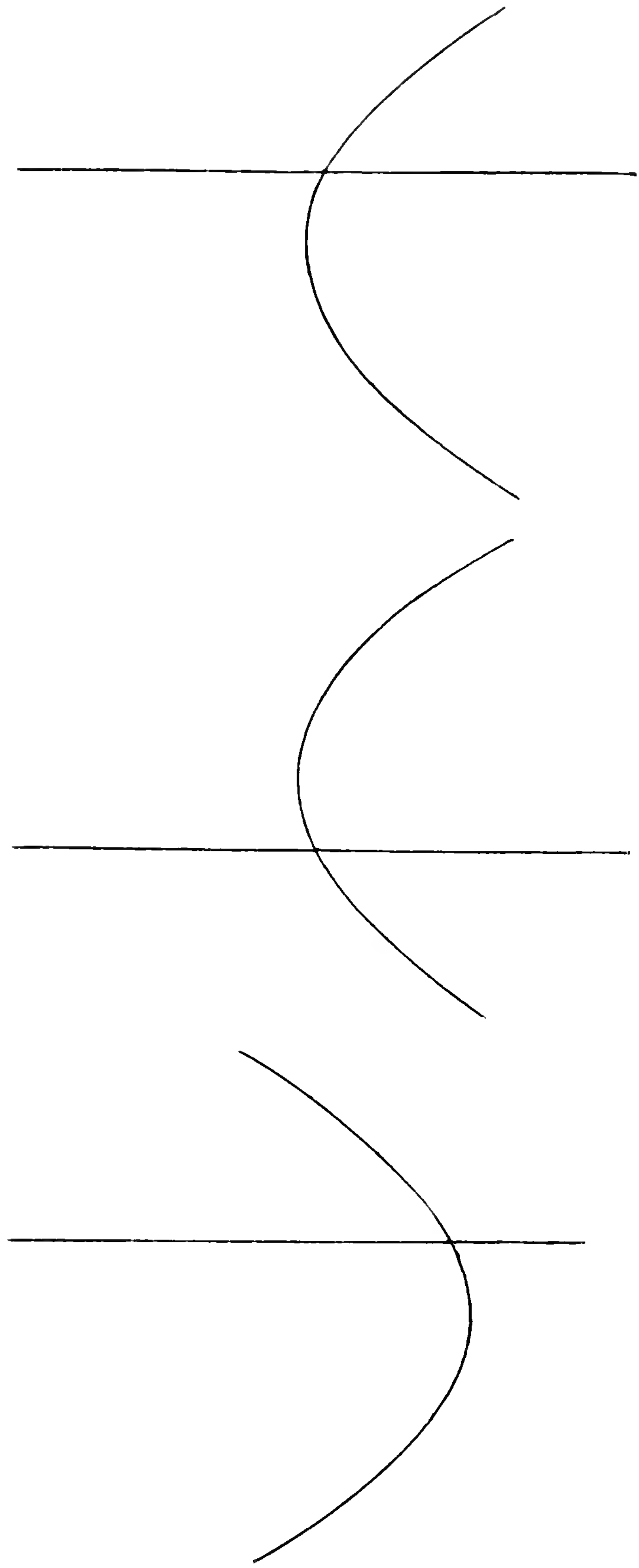


FIG. 68.

Observe, with regard to slope, that when the curve slopes upward where it cuts the vertical axis, as in fig. 68 i. or ii., the coefficient b, which measures the slope there, is positive; but when it slopes downward as in fig. 68 iii. the term involving b is negative. If the curve cut the vertical axis without slope, or horizontally, the term involving b would disappear, and in such a case the parabola would be represented by $y = a + cx^2$ (fig. 69).

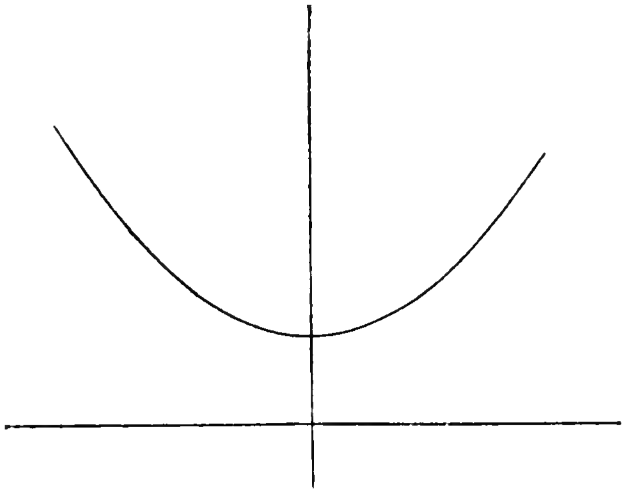


FIG. 69.

If the curve cut off no length on the axis of y, that would be indicated by the non-existence of the constant a, so that that parabola would be written $y = cx^2$ (fig. 70),

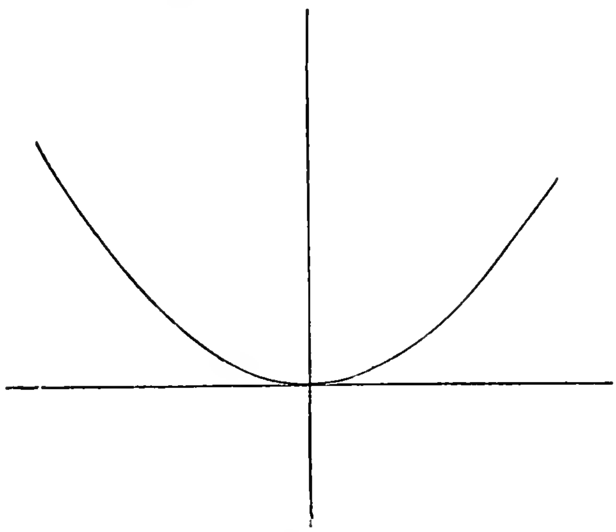


FIG. 70.

and this is the simplest expression for a parabola possible.

Observe that it is the same curve all the time; it is only shifted with respect to the axes by the different values which can be attributed to the constants.

c is the curvature term, and when c is positive it curves upwards, like 68 i.; when c is negative it curves downwards, like 68 ii. or iii.

$y = -cx^2$ would be like this (fig. 71):

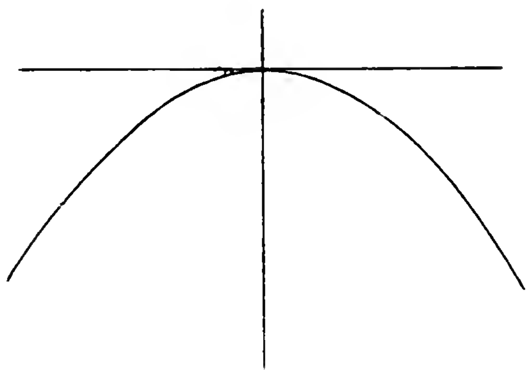


FIG. 71.

If we want the parabola to look like this (fig. 72):

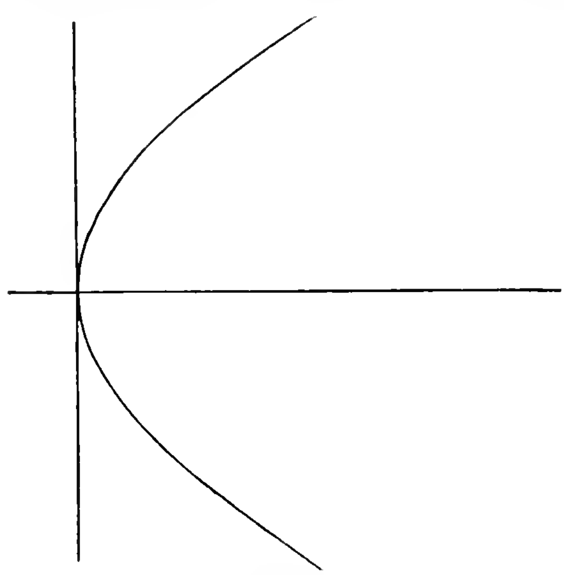


FIG. 72.

we have only to turn it through a right-angle, that is to say, interchange the axes of x and y, and write

$$x = cy^2,$$

or if we wrote $x = -cy^2$ it would look like this:

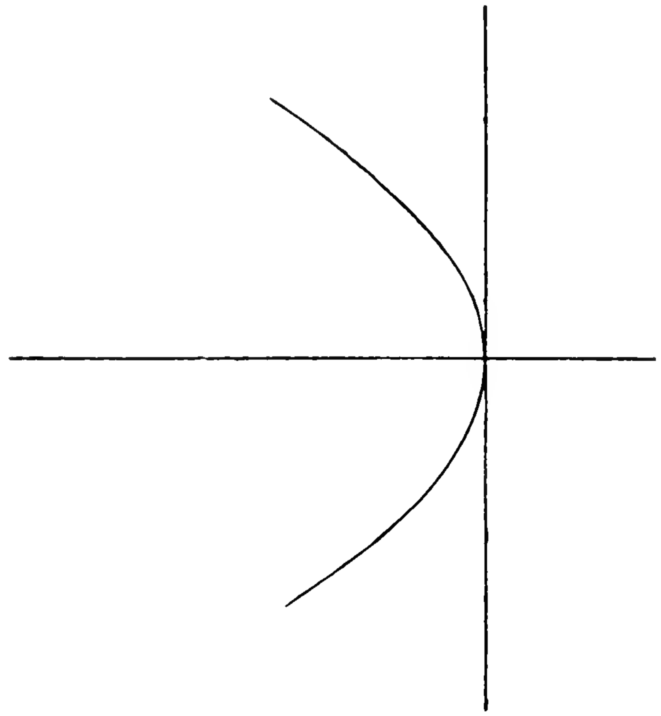


FIG. 73.

If we wrote $x = a + cy^2$ it would become like this:

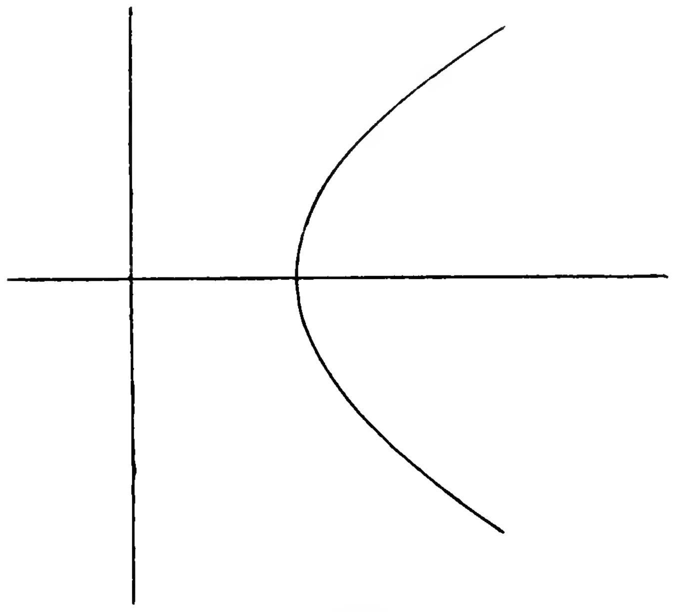


FIG. 74.

where the x intercept is a.

If we wrote $x = a + by + cy^2$ it would slope up where it cuts the axis.

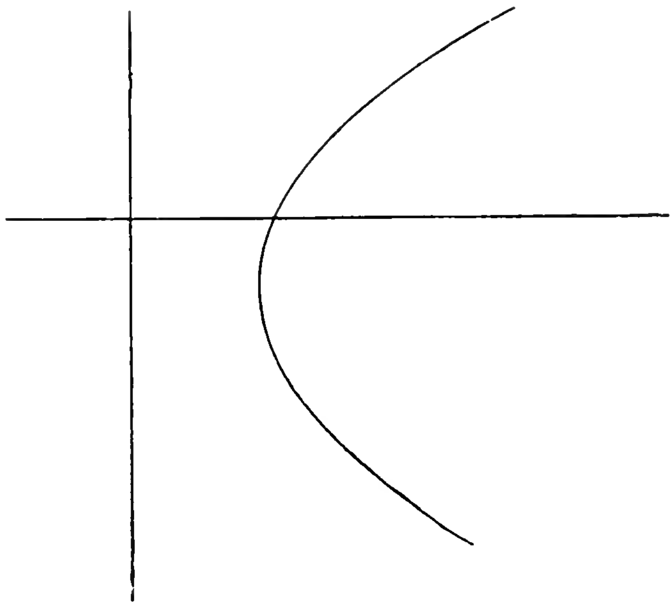


FIG. 75.

If we had $x = a - by + cy^2$ it would slope down at the foot, like this:

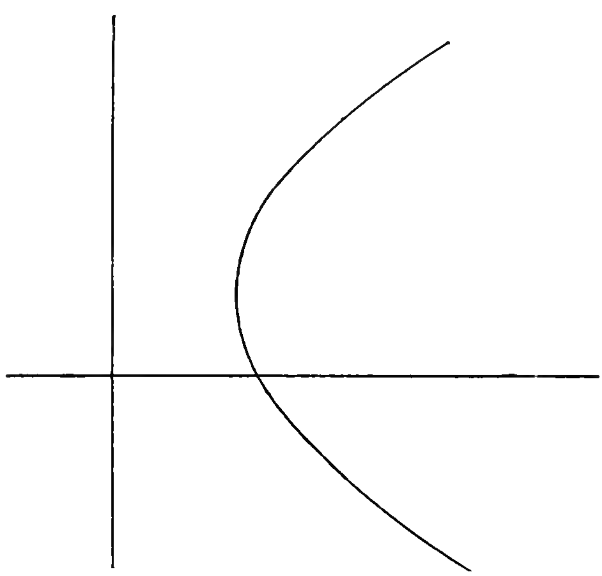


FIG. 76.

If $x = -a + cy^2$ the curve would cut the x axis on the left:

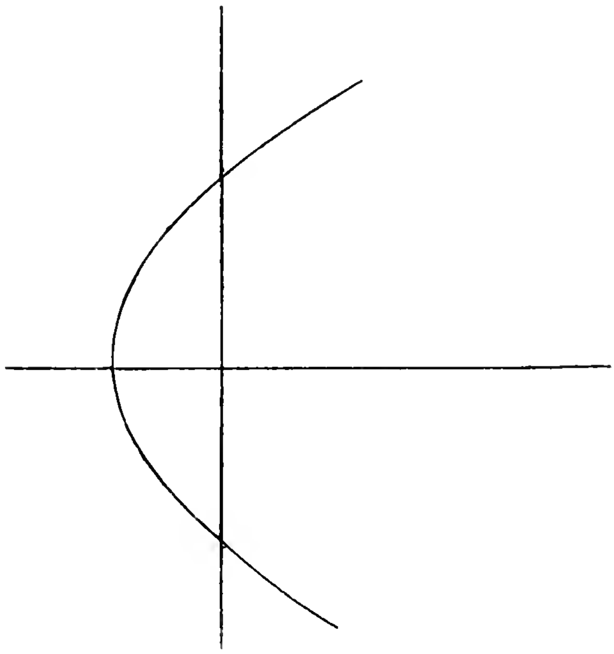


FIG. 77.

If we introduce a term containing x^2 as well as a term containing y^2, and if necessary a term xy, we can tilt the parabola in any desired direction and place it anywhere in the plane: though in these cases there is a risk that it may cease to be a parabola; and if we introduced a term x^3 or y^3 its parabolic character is bound to be spoilt; just as the introduction of either x^2 or y^2 destroyed the straightness of the line $y = a + bx$.

The next step towards algebraic complexity is what is called the **cubic parabola** $y = a + bx + cx^2 + dx^3$,
where again the coefficients or constants, a, b, c, d, may be any of them positive or any of them negative.

The beginner can plot this and see what it looks like,—proceeding after the same fashion as before; that is attributing any arbitrary value, positive or negative, to the four constants *abcd*, and then reckoning the value of y for different values of x; afterwards plotting them to any convenient scale—remembering that the horizontal scale and the vertical scale need not necessarily be the same, but may be chosen independently, to suit the convenience of the draughtsman.

Inverse Variation.

It frequently happens that two quantities are connected in such a way that one increases when the other decreases. For instance, the plentifulness of a commodity, say corn, and its price. In a year of good harvest the price of wheat drops. During a famine the price rises. It might happen that the total money to be obtained, for the produce of a certain farm acreage, was constant, whether the crop was plentiful or sparse. Such a simple relation as that is not likely to hold exactly; but if it did, the two quantities—the price and the supply—would be said to vary inversely as one another, that is to vary in **inverse** simple proportion, so that their **product** remained constant; whereas if they varied in **direct** simple proportion it would be their **ratio** which remained constant.

It does not follow that this law of simple inverse variation holds because one quantity decreases and the other increases; all manner of complicated relations may hold between such quantities; the law of inverse proportion is the simplest possible, and there are a great many cases where it holds, or holds very approximately.

Take a piece of india-rubber cord or tubing, and pull it out longitudinally; as the length increases, the sectional area diminishes, and it is a matter of measurement to ascertain what relation holds between these two things.

If the tube were filled with water and were then pulled out, the behaviour of the water would furnish a test of how the sectional area varied with the length. If the water continued to fill, or to stand at the same level in, the tube (which might terminate at one end in a piece of glass tubing, for convenience of observation, and be closed with a solid plug at the other), that would mean that the sectional area and the length varied inversely as one another; in other words that their product was constant; for the product of length and sectional area is the volume, and it is the volume which the water measures. Try the experiment. As a matter of fact the water will be found to sink a little as the tube is stretched, showing that the volume increases slightly: the law of simple inverse proportion does not hold in this case.

Consider another case then. Take a vessel of variable capacity, for instance a cylinder and piston; or a tube open at one end, which can be plunged mouth downward under a liquid, like a long diving bell, and can be lowered or raised so that the air in it shall be compressed or expanded at pleasure.

If a pressure gauge is attached, it will be possible to read how the pressure increases as the volume diminishes; and it will be found that the two vary inversely as one another, provided one is careful to take the whole dry-air pressure and the whole volume into account. The product of pressure and volume will be found experimentally constant; because if one is halved, the other will be doubled; if one is trebled, the other becomes one-third of its original value; and so on, a law which is written:

$$p \propto \frac{1}{v},$$

$$\text{or } v \propto \frac{1}{p}, \qquad \text{or } pv = \text{constant}.$$

These are all statements of the same fact; p standing for the pressure, and v for the volume of a given quantity of dry air at constant temperature. If the vessel leaks, so that the actual amount of air under observation changes, the law will not apply. Nor will it hold if the temperature is allowed to change. For if air is warmed it expands, that is, it increases in volume or in pressure or in both together; and there is no necessity at all for the pressure to decrease as the volume increases unless the temperature is maintained constant. Hence a complete statement is that pv = constant if T is constant; or, if we choose so to express it, pv = const. provided $dT = 0$, *i.e.* provided the difference of T is zero, which is only another way of saying "if T is constant."

Suppose, subject to $dT = 0$, the pressure was increased by a small amount dp, and the volume thereby decreased a small amount $-dv$, we should have the new product of pressure and volume expressed thus:

$$p'v' = (p+dp)(v+dv),$$

and this product must equal the old product, pv, because of the law that the product of pressure and volume is constant; so, multiplying out, we get

$$pv+p\,dv+v\,dp+dp\,dv = pv.$$

Wherefore, ignoring the second-order small quantity $dpdv$, we have

$$v\,dp+p\,dv = 0$$

or

$$\frac{dp}{p} = -\frac{dv}{v}$$

as another statement of the law of inverse variation.

Summary.

DIRECT VARIATION

$$y = kx,$$

$$\text{or } dy = kdx,$$

$$\text{or } \frac{dy}{dx} = k = \frac{y}{x},$$

$$\text{or } xdy - ydx = 0,$$

$$\text{or } \frac{dy}{y} - \frac{dx}{x} = 0.$$

Inverse Variation.

$$xy = k,$$

$$\text{or } xdy + ydx = 0,$$

$$\text{or } \frac{dy}{dx} = -\frac{y}{x} = -\frac{k}{x^2},$$

$$\text{or } \frac{dy}{y} + \frac{dx}{x} = 0.$$

So that if $\frac{dy}{dx} = \pm\frac{y}{x}$, that is if the ratio of differences is numerically the same as the ratio of the quantities themselves, it is a case of simple proportion; but two distinct cases are given by the alternative sign :

if the sign is + it is direct proportion ;
if the sign is − it is inverse proportion.

CHAPTER XL.

Pumps and Leaks.

When you pump water out of a reservoir, taking a barrel full of water out at each stroke, the quantity of water remaining decreases in an arithmetical progression, of which the first term was the contents of the well, and the common difference is the contents of the pump barrel. If one were called V, the other v (read big V and little v), the level in the well would fall after successive strokes in the following series:

$$h_0, h_1, h_2, \dots h_n,$$

where h_0 is the height of the water before pumping began, and h_n is the height after n strokes of the pump,

such that $$\frac{h_0}{V} = \frac{h_1}{V - v} = \frac{h_2}{V - 2v} = \frac{h_3}{V - 3v} = \dots,$$

a mode of writing which is called a continued proportion.

The quantity of water remaining in the well descends in a decreasing Arithmetical Progression,

$$V, V - v, V - 2v, \text{ etc.} \dots V - nv,$$

and the well is empty when $nv = V$; or the number of strokes required to empty it is the ratio of the capacities concerned, V/v.

The height or level of the water in the well goes in the same sort of progression, and h_n is zero after V/v strokes.

But now consider an air pump instead of a water pump. The peculiarity of air or any other gas is that it always fills the vessel which contains it, and does not accumulate in one part as a liquid does. A bottle may be said to be "full" of air, whether it contains much or little, in the sense that all parts are equally full. It is always full in this sense, and it can never be full in any other sense; because however much air is in, some more can always be pumped in: the only limit is the bursting and destruction of the bottle. Or, if it were made of porous material, it could be said to be as full as it would hold when the rate of leak was equal to the rate at which air was being pumped in; but even that could be exceeded by beginning to pump a little faster.

With a liquid, on the other hand, a bottle may be properly said to be "half-full"; it can also be completely full, for you cannot pump more than a certain quantity into a closed vessel. If it is an open vessel the rate of leak at a certain definite point becomes suddenly equal to the rate of supply, and the vessel overflows; which is a good practical method for maintaining a constant level.

There is no such easy method for providing a constant air or gas or steam pressure, though something of the kind is attempted by means of a leak so adjusted as to suddenly change from near zero to something considerable, at a certain critical pressure,—such an arrangement being called a "safety-valve." Locomotive boilers are usually filled with steam to this pressure before a train starts on a long journey, and any excess steam which the furnace generates blows off noisily in a visible cloud.

If you were to pump air *into* a closed chamber, a barrel full of atmospheric air would be injected at every stroke, and the pressure would rise in an increasing arithmetical progression,

$$p_0,\ p_1,\ p_2,\ \cdots p_n,$$

p_0 being the initial pressure before pumping, and p_n the pressure after n strokes,

such that $$\frac{p_0}{V} = \frac{p_1}{V+v} = \frac{p_2}{V+2v} = \dots = \frac{p_n}{V+nv},$$

the pressure being proportional to, and a measure of, the extra quantity of air injected. But if a pump is used to *eject* the air, that is to say, to draw *out* from a closed chamber a barrel full of air at every stroke, the law of decreasing pressure is different: it then forms a geometrical progression.

For the same quantity of air is not removed each time. The same **volume** is removed, but it is removed from air of gradually diminishing density. The air keeps on getting rarefied, and this rarefied air it is which has to supply the pump barrel; so that during every direct stroke the air which at first occupied V expands to occupy $V+v$, and then the excess is ejected into the atmosphere at the return stroke of the pump, ready for the expanding operation to begin again.

Thus, assuming the temperature to remain constant, we have the pressure diminished at every stroke in the constant ratio $\frac{V}{V+v}$; and the series $p_0, p_1, p_2, \dots p_n$ is a decreasing geometric progression with the common ratio $V/(V+v)$.

So that
$$p_1 = \frac{V}{V+v}p_0,$$
$$p_2 = \frac{V}{V+v}p_1 = \left(\frac{V}{V+v}\right)^2 p_0,$$
$$p_3 = \frac{V}{V+v}p_2 = \left(\frac{V}{V+v}\right)^3 p_0,$$
.... etc.,

the ratio of the pressure at beginning and end of any stroke being constant, viz.

$$\frac{p_n}{p_{n-1}} = \frac{V}{V+v}, = \frac{1}{r}, \text{ say; and } p_n = p_0 r^{-n}.$$

Hence the operation of an ordinary exhausting air pump is governed by the law of a decreasing geometrical progression; and an infinite number of strokes would be necessary completely to empty the vessel, that is to reduce the pressure to zero.

Leaks and Compound Interest.

Now suppose instead of being pumped out the vessel were full of compressed air and were to leak; or suppose there were a cistern full of water with a crack in the bottom of it; the pressure in the one case and the level in the other would fall according to a certain law. If the leakage rate were constant, that is to say if the same amount of material escaped every second, the law would be a decreasing A.P.; but that is never the case in fact. The size and circumstance of the leak-orifice being constant, the amount of matter which escapes through it depends on the force with which it is urged, that is to say on the pressure behind it. A high pressure reservoir, or a tall full cistern, would leak fast, the air or water rushing out of the leak with violence; whereas towards the end, when the vessel was nearly empty, the rush would have degenerated into a mere dribble or ooze, unless of course it had worn the hole larger—which we will not suppose to be the case. With a constant sized orifice the rate of leak is therefore proportional to that which causes the leak, viz. the pressure; and so the pressure keeps on falling, at a rate depending on itself: a curious and important, because, in one form or another, a frequent case.

When you come to think, it is just the compound-interest case, but inverted. Capital increases at a rate proportional to itself: when small it grows slowly, that is by small additions, when large it grows quickly. If we call the capital at any moment x, its rate of growth will be $\frac{dx}{dt}$, since dx means an increase of capital, and dt the time during which this increase occurs. In the case of capital the increase is somewhat discontinuous: the interest is added every year, or it may be every month, or perhaps every day, but not every instant. Let us assume that it is continuous however, so that it increases from moment to moment at the rate $\frac{dx}{dt}$; this rate of increase will be proportional to x itself, and of course to the percentage which is granted.

Suppose for instance it was 5 per cent., or ·05, the law of increase would be

$$\frac{dx}{dt} = \cdot 05x;$$

the interest, dx, is proportional to the rate, ·05, to the capital on which it is paid, x, and to the time during which it has accumulated, dt; or $dx = \cdot 05x\,dt$.

If it were 4 per cent., or 3 per cent., or $2\frac{1}{2}$ per cent., of course we should substitute ·04, or ·03, or ·025, for the ·05 numerical coefficient.

So with the leak, we have similarly to express that $-dx$, the loss of pressure, is proportional to the pressure, and to the time, and to a leak-aperture constant which we will call k; so

$$dx = -kx\,dt;$$

for to express that it is a loss and not a gain, a decrease not an increase, we must apply to it a negative sign. The x might be pressure, or it might be level or "head,"—the two are proportional in the case of water; but level has no meaning in the case of gas, so we will take "pressure" as the more general term, and, denoting it by p instead of x, write the law of leak, in the simplest possible case of a constant orifice, as

$$\frac{dp}{dt} = -kp;$$

the rate of fall of pressure is proportional to the pressure from instant to instant, diminishes as it diminishes, and does not reach zero till the pressure reaches zero. The pressure in fact decreases as a geometric progression. But it is a geometric progression with one curious feature about it, it is continuous, not discontinuous like numbers; it does not go in steps or jumps, like compound interest added every year or every day, but it is like compound interest added or rather subtracted every instant, with complete continuity, according to a smooth curve, the logarithmic or G.P. curve, see page 357 or 101 or 179.

Cooling.

The cooling of a hot body under simplest conditions follows just the same law; the rate of fall of temperature is proportional to the actual excess of temperature above surrounding objects. If we denote this temperature by θ and time by t,

$$\frac{d\theta}{dt} = -k\theta$$

expresses the simplest possible law of cooling as the heat escapes or leaks from the body into surrounding air or space.

It is instructive to put a thermometer into a flask or pan of very hot water, and read the thermometer from time to time; at first every half-minute, or oftener, then every minute, and then as it cools more and more slowly, it will suffice to read it every five minutes; finally plotting the result thus:

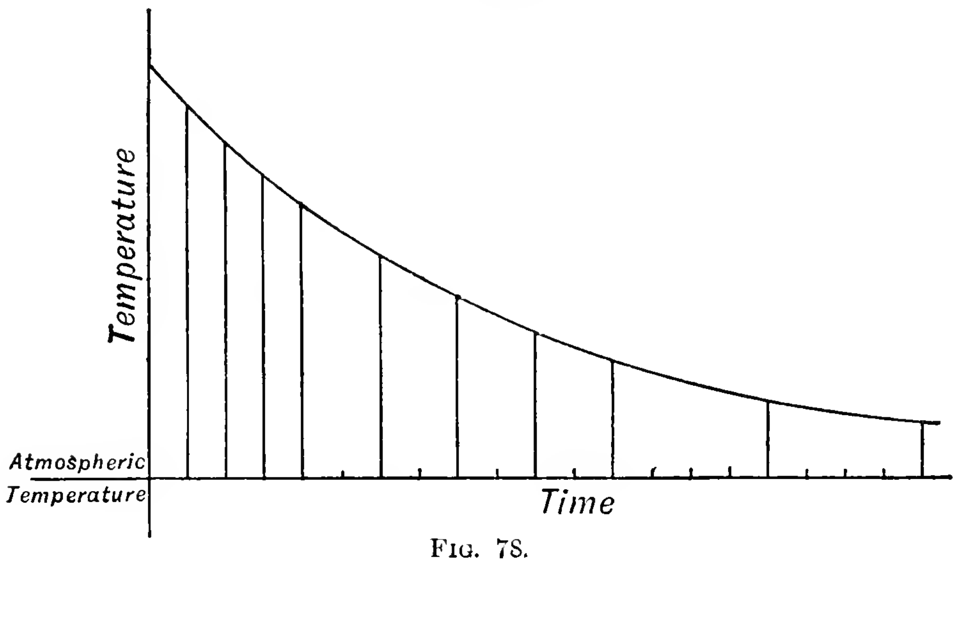


FIG. 78.

By choosing different vessels, say one black and one bright, or by choosing similar vessels and filling them with different liquids, one water and another turpentine say, many instructive observations can be made; but a discussion of these would carry us too far at present.

The curve of cooling is identical with the curve of leaking; and the curve of leaking might be plotted by reading a pressure gauge, or by reading the level of a leaking water-reservoir, from time to time. And both are curves of decreasing G.P. or are logarithmic curves.

Electric Leakage.

Experiments in electricity are more difficult, but if it were possible to read satisfactorily by means of an electrometer the potential of an electrified body or Leyden jar or condenser which was steadily leaking, it would be found to obey the same law.

Continuously decreasing G.P.

Now see how to express a quantity which decreases geometrically with perfect continuity, and not by steps, as time goes on. Notice that **time** is a continuous progression; there is no means of hurrying it; one day is like another, and they follow on with absolute regularity. Time is an inexorable arithmetical progression, an increasing one if you think of your birth, a decreasing one if you attend to the other end of your life. Whatever can be varied, time cannot: at least not by us.

Now when a vessel is leaking, the pressure is to be multiplied by a constant factor, some proper fraction, at each successive equal interval of time.

So p_0 at the start, when time is 0, or say at 12 o'clock noon, becomes let us suppose $\frac{1}{2}p_0 = p_1$ after the lapse of 1 hour, or at 1 p.m. If so, then in another hour it will have fallen to $\frac{1}{2}p_1$, or what is the same thing $\frac{1}{4}p_0$; in yet another hour, that is at 3 o'clock, it will be $\frac{1}{8}p_0$, and in n hours it will be $\frac{1}{2^n}p_0$; hence it will have fallen continuously down the decreasing geometrical-progression-curve depicted on page 101.

But why should we suppose it *halved* in each unit of time? We can be more general than that, and say that it is reduced to $\frac{1}{r}p_0$ (read, 'one r-th of p-nought') after the lapse of one hour, where r is some number greater than unity; then in another hour the pressure will have become $\frac{1}{r}p_1$, or what is the same thing $\frac{1}{r^2}p_0$.

So the pressure at 2 p.m. is $p_2 = r^{-2}p_0$,

at 3 p.m. $p_3 = r^{-3}p_0$,

and at n hours after noon $p_n = r^{-n}p_0$.

Or we might say that, at any time t after the start, the pressure is $p = r^{-t}p_0$.

This then is the law:

$$p = p_0 r^{-t},$$

or $$\frac{p}{p_0} = r^{-t},$$

or $$\log\frac{p}{p_0} = -t\log r,$$

or $$\log p_0 - \log p = t\log r,$$

or $$\log r = \frac{\log p_0 - \log p}{t},$$

all expressive of the very same fact.

Now r is a constant depending on the size of the opening the viscosity of the escaping fluid (or on the covering and contents of the cooling body), and any other circumstance which can affect the rate of leak, other than pressure (or temperature) and time. And the log of r is the diminution of the log of the pressure, during any lapse of time, divided by the time which has elapsed. It is the ratio of the logarithmic diminution, or decrement, to the time; it is the decrement of the logarithm of the pressure per unit time, and is technically known as the "logarithmic decrement" of the pressure (or of the temperature in the case of a cooling body, or of the potential in the case of an electric leak, or of the level in the case of a leaking cistern).

To measure $\log r$, all we have to do is to read the pressure (or temperature, etc.) at any one instant, and then read it again some time later, observing the interval of time.

Let the two readings be denoted by p_0 and p_n, and let the intervening time be n seconds, then

$$\frac{\log p_0 - \log p_n}{n}$$

is the logarithmic decrement per second, and is a measure of the constant we have called $\log r$.

Thus the law which was at first expressed in differential form as

$$\frac{dp}{dt} = -kp \quad \text{or} \quad \frac{dp}{p} = -k\,dt,$$

can also be expressed in integral form as

$$p = p_0 r^{-t} \quad \text{or} \quad \log p_0 - \log p = t \log r;$$

and it now becomes necessary to ascertain and express the relation between the two constants k and r, which evidently refer to the same sort of thing, viz. the fixed circumstances of the leak.

Now remembering what we know of exponentials, let us see if we can puzzle out the connexion between these constants.

The law that we have written expresses the fact that pressure decreases geometrically as time increases arithmetically: a constant factor is characteristic of one progression, while a constant difference characterises the other.

We know that if p_0 is the pressure at the era of reckoning, that is at the instant from which time is to be reckoned, then at any time t the pressure is $p = p_0 r^{-t}$, and at any other time t' the pressure is $p' = p_0 r^{-t'}$, therefore

$$\frac{p'}{p} = r^{-(t'-t)}.$$

Now let the change be small, so that $p' - p = dp$ and $t' - t = dt$; then the last equation is

$$\frac{p + dp}{p} = 1 + \frac{dp}{p} = r^{-dt},$$

or
$$\frac{dp}{p} = r^{-dt} - 1 = -dt \,.\, \log r.$$

The last step we are not supposed to know enough yet to justify; but, assuming it and deferring its justification to page 425, we see that

$$\frac{dp}{p} = -\log r \,.\, dt,$$

or
$$\frac{dp}{dt} = -p \log r;$$

and this we can compare with the equation at which we started (p. 405),

$$\frac{dp}{dt} = -kp.$$

Thus the relation between the constants k and r is simply

$$k = \log r,$$

and accordingly k is itself the logarithmic decrement of the pressure per second.

Summary.

The physical meaning of k is $-\frac{1}{p}\frac{dp}{dt}$,

the physical meaning of $\log r$ is $\frac{\log p - \log p'}{t}$.

These two things turn out to be mathematically the same thing.

But when $\log r$ is thus written, what base is intended for the logarithm? There is nothing to say that the base is 10, and indeed no explicit assertion has been made about any base whatever: all that has been asserted is that p is to be a quantity whose rate of change is proportional to itself, or equal to itself when multiplied by the constant k or $\log r$.

There is evidently something here worth investigation from the purely mathematical point of view. It is a definite mathematical question to put "What is that quantity whose rate of change shall be proportional to itself? how is such a quantity to be expressed in general?" To investigate this question, we can study the rates of variation of various algebraic expressions.

CHAPTER XLI.

Differentiation.

TAKE the area of a square, and ask how it varies with the side which contains it, when the square slightly expands. We already know, but we will go through the process, especially for a very small or infinitesimal increment of the side. Let the side be x, and the area of the square be called y, so that $y = x^2$, then when the square is warmed a little, x increases by the amount dx, and y increases by the amount dy, such that

$$\begin{aligned} y + dy &= (x + dx)^2 \\ &= x^2 + 2x dx + (dx)^2. \end{aligned}$$

Now let dx be so small that the square of dx may be utterly neglected in comparison with dx itself. In the limit suppose dx actually infinitesimal, so that $(dx)^2$, being $dx \times dx$, is again or still further infinitesimal, even in comparison with dx; then

$$y + dy = x^2 + 2x dx;$$

but $y = x^2$, therefore, subtracting, there remains $dy = 2x dx$,

or $$\frac{dy}{dx} = 2x;$$

whence the rate of change of area of an expanding square, per unit expansion of edge, is twice the length of one of the sides: a very elementary statement, but not obvious. It is of course a general analytic or algebraic result, and in no way depends upon any geometrical meaning attached to y^2. The geometrical square is only a special case, and it is convenient as an illustration; but it would be equally true for any other variation of one quantity as the square of another; for instance, the relation between the velocity of a falling body and the height it has fallen, so well known in mechanics, is written $v^2 = 2gh$, and this we can re-write in differential form

$$d(v^2) = 2vdv = 2gdh,$$

or

$$\frac{dv}{dh} = \frac{g}{v},$$

which gives us the extra speed gained for each additional foot or centimetre or other small unit of height.

Suppose for instance the height already fallen were 100 feet: a dropped stone would have acquired a speed of 80 feet a second. By the time it has dropped a foot more, the above equation asserts that its speed will have increased by the amount $\frac{32}{80} = \frac{2}{5} = {\cdot}4$ feet per sec.

We might also get the above relation thus:

$$v^2 = 2gh,$$

$$v'^2 = 2g(h+1);$$

$$\therefore\ v'^2 - v^2 = 2g,$$

or

$$v' - v = \frac{2g}{v' + v} \bumpeq \frac{2g}{2v} = \frac{g}{v};$$

but in this case there is an approximation, because 1 foot added to 100 is by no means infinitesimal though it is moderately small. Consequently a sort of average or mean has to be taken between v and v', which in the limit would be ultimately equal.

The expression $dy = 2xdx$ we long ago illustrated by the two strips, each equal to xdx in area, which went to form the increase of surface in a square plate x^2 expanded by heat (page 369); the little corner bit $(dx)^2$ being ignored, because when the strips themselves are infinitesimal, the infinitesimal bit of each at the ends is nought in comparison, or is said to be an infinitesimal of the second order.

Similarly we may deal with the expansion or variation of a cubical block of side x.

Denote its volume by $y = x^3$,
then when it expands infinitesimally

$$y + dy = (x + dx)^3$$
$$= x^3 + 3x^2 dx + \text{infinitely smaller quantities};$$
$$\therefore \ dy = 3x^2 dx,$$

or $$\frac{dy}{dx} = 3x^2,$$

or the rate of expansion of a cubical volume, per unit increase of a side, is three times the area of one of its faces.

Observe that the rate of increase of an area is a length, while that of a volume is an area; but that is because the rate of increase is taken per unit of length. If it were taken per unit of time or of temperature, and if, as before, we write $y = x^3$, we could say that

$$\frac{dy}{dt} = 3x^2 \frac{dx}{dt},$$

or the rate of variation of the volume with respect to any outside or independent variable, such as temperature, $\left(\frac{dy}{dt}\right)$ (*i.e.* the cubical expansion per degree), is greater than the rate at which each edge expands for the same variable, $\left(\frac{dx}{dt}\right)$ (the linear expansion per degree), in the ratio of three times the area of one of its faces.

Stated thus it is perhaps hardly geometrically evident, nor need it be made so. What is geometrically capable of illustration is the fact that

$$dy \text{ or } d(x^3) = 3x^2 dx.$$

Other expressions of the same kind of fact are best treated as mere analytic or algebraic statements, without any *necessary* geometrical signification.

So we learn that to get the small change of any quantity we have only to attend to the early terms of a binomial expansion: two only, if the change is infinitesimal.

For instance, to find $d(x^4)$, that is to express it in terms of dx, we let x increase by dx, and then expand and neglect all beyond the first power of dx; thus

$$(x+dx)^4 = x^4 + 4x^3 dx + \text{higher powers};$$

but $\quad x + dx = x'$, and $d(x^4)$ *means* $x'^4 - x^4$,

therefore $\quad d(x^4) = 4x^3 dx.$

Similarly $\quad d(x^5) = 5x^4 dx,$

$$d(x^6) = 6x^5 dx, \text{ and so on, until}$$

$$d(x^{12}) = 12x^{11} dx,$$

and $\quad dx^n = nx^{n-1} dx.$

So also with fractional indices:

For instance, to find $d\sqrt{x}$. Expand, and ignore all higher powers of the infinitesimal quantity dx.

$$(x+dx)^{\frac{1}{2}} = x^{\frac{1}{2}} + \tfrac{1}{2}x^{-\frac{1}{2}}dx + \text{higher powers},$$

but $\quad (x+dx)^{\frac{1}{2}} = \sqrt{x'},$

so $\quad d\sqrt{x} = \sqrt{x'} - \sqrt{x} = \tfrac{1}{2}x^{-\frac{1}{2}}dx = \dfrac{dx}{2\sqrt{x}}.$

Again $\quad (x+dx)^{\frac{3}{2}} = x^{\frac{3}{2}} + \tfrac{3}{2}x^{\frac{1}{2}}dx,$

therefore $\quad d\sqrt{x^3} = dx^{\frac{3}{2}} = \tfrac{3}{2}x^{\frac{1}{2}}dx = \tfrac{3}{2}\sqrt{x}\,.\,dx.$

Or take negative indices:

To find $d\left(\dfrac{1}{x}\right)$ or dx^{-1}; expand again.

$$(x+dx)^{-1} = x^{-1} - x^{-2}dx,$$

but $$(x + dx)^{-1} = \frac{1}{x'},$$

so $$d\left(\frac{1}{x}\right) = \frac{1}{x'} - \frac{1}{x} = -x^{-2}dx = -\frac{dx}{x^2}.$$

And in general, whatever n may be,

$$\mathbf{dx^n = nx^{n-1}dx},$$

a perfectly general result, worth thoroughly learning and applying to special cases.

Even in the case when $n = 0$ it holds good; for then it says that $$dx^0 = 0,$$
which we know is true, because $x^0 = 1 =$ constant, and so its differences or variations must be zero.

If $n = 1$ it gives $dx = 1 \,.\, dx$, which is a mere identity.

If $n = 2$, it gives $dx^2 = 2xdx$;

if $n = 3$, ,, $dx^3 = 3x^2dx$,

which we separately verified; and so on.

Examples.

Check the following statements:—

$dx^7 = 7x^6dx$; $\quad \frac{dx^7}{dx} = 7x^6$; $\quad \frac{dx^3}{dx} = 3x^2$;

$dax = adx$; $\quad d(ax^2) = 2axdx$; $\quad dax^3 = 3ax^2dx$.

$\frac{dax^3}{dx} = 3ax^2$; $\quad \frac{dby^4}{dy} = 4by^3$; $\quad \frac{d5x^3}{dx} = 15x^2$;

$\frac{d}{dx}(\frac{1}{2}x^2) = x$; $\quad \frac{d}{dx}(\frac{1}{6}x^3) = \frac{1}{2}x^2$; $\quad \frac{d}{dx}(\frac{1}{24}x^4) = \frac{1}{6}x^3$;

$\frac{d7x^5}{dx} = 35x^4$; $\quad d(x+y) = dx + dy$; $\quad d(ax + by) = adx + bdy$;

$d(ax + bx^2) = adx + 2bxdx = (a + 2bx)dx$;

$\frac{d}{dy}(ay + by^2) = a + 2by$; $\quad d(a + bx) = bdx$;

$$\frac{d}{dx}(a+bx) = b; \qquad d(a+bx+cx^2) = (b+2cx)\,dx;$$

$$\frac{d}{dx}(a+bx+cx^2+x^3) = b+2cx+3x^2; \qquad \frac{d}{dx}(A+Bx^n) = nBx^{n-1};$$

$$\frac{d}{dx}(A+Bx+Cx^2+Dx^3+Ex^4+\ldots+Zx^n)$$
$$= B+2Cx+3Dx^2+4Ex^3+\ldots+nZx^{n-1};$$

$$d\left(\frac{1}{x}\right) = -\frac{dx}{x^2}; \qquad \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2};$$

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}; \qquad \frac{d}{dx}\left(\frac{1}{x^3}\right) = -\frac{3}{x^4};$$

$$\frac{d}{dx}\left(\frac{A}{x}\right) = -\frac{A}{x^2}; \qquad \frac{d}{dx}\left(\frac{A}{x}+B\right) = -\frac{A}{x^2};$$

$$\frac{d}{dx}\left(\frac{a}{bx}\right) = -\frac{a}{bx^2}; \qquad \frac{d}{dx}\left(\frac{A}{x}+B+Cx\right) = -\frac{A}{x^2}+C;$$

$$\frac{d}{dx}\left(\frac{A}{x^2}+\frac{B}{x}+C+Dx+Ex^2\right) = -\frac{2A}{x^3}-\frac{B}{x^2}+D+2Ex;$$

$$dx^{\frac{1}{2}} = \tfrac{1}{2}x^{-\frac{1}{2}}\,dx; \qquad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}};$$

$$\frac{d}{dx}(a\sqrt{x}) = \frac{a}{2\sqrt{x}}; \qquad \frac{d}{dx}(a+b\sqrt{x}) = \frac{b}{2\sqrt{x}};$$

$$dx^{-\frac{1}{2}} = -\tfrac{1}{2}x^{-\frac{3}{2}}\,dx; \qquad \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = -\frac{1}{2(\sqrt{x})^3};$$

$$dx^{\frac{3}{2}} = \tfrac{3}{2}x^{\frac{1}{2}}\,dx; \qquad \frac{d}{dx}(\sqrt{x})^3 = \tfrac{3}{2}\sqrt{x};$$

$$\frac{d}{dx}\left(a\sqrt{x}+b+\frac{c}{\sqrt{x}}\right) = \frac{a}{2\sqrt{x}}-\frac{c}{2(\sqrt{x})^3} = \frac{1}{2\sqrt{x}}\left(a-\frac{c}{x}\right);$$

$$\frac{d}{dx}(a+bx)^2 = \frac{d}{dx}(a+2abx+b^2x^2) = 2ab+2b^2x = 2b(a+bx);$$

$$\frac{d}{dx}\left(\frac{a}{x}+bx\right)^2 = \frac{d}{dx}\left(\frac{a^2}{x^2}+2ab+b^2x^2\right) = -\frac{2a^2}{x^3}+2b^2x$$
$$= 2\left(\frac{a}{x}+bx\right)\left(b-\frac{a}{x^2}\right);$$

$$\frac{d}{dx}\left(a\sqrt{x}+\frac{b}{\sqrt{x}}\right)^2 = \frac{d}{dx}\left(a^2x+2ab+\frac{b^2}{x}\right) = a^2-\frac{b^2}{x^2};$$

$$d(v^2) = 2vdv; \qquad d(2gh) = 2gdh; \qquad \frac{d}{dt}(v^2) = 2v\frac{dv}{dt};$$

$$\frac{d}{dh}v^2 = 2v\frac{dv}{dh}; \qquad \frac{d}{dh}(2gh) = 2g; \qquad \frac{d}{dt}(2gh) = 2g\frac{dh}{dt};$$

$$\frac{d}{du}(au^2+bu+c) = 2au+b; \qquad \frac{d}{du}av^2 = 2av\frac{dv}{du};$$

$$\frac{d}{dt}(au^2+bu+c) = 2au\frac{du}{dt}+b\frac{du}{dt} = (2au+b)\frac{du}{dt};$$

$$\frac{d}{dt}\left(Ax^2+Bx+C+\frac{D}{x}\right) = \left(2Ax+B-\frac{D}{x^2}\right)\frac{dx}{dt}.$$

CHAPTER XLII.

A Peculiar Series.

WE are now able to write down a set of algebraic terms, each of which shall be the differential-coefficient of the one following it:

$$0 + 1 + x + \tfrac{1}{2}x^2.$$

Of this we might make a regular series, for just as $\frac{1}{2}x^2$ differentiated gives x, so $\frac{1}{3}x^3$ differentiated would give x^2, and $\dfrac{x^3}{2\,.\,3}$ would give $\frac{1}{2}x^2$. So also $\dfrac{x^4}{2\,.\,3\,.\,4}$ differentiated would give $\dfrac{x^3}{2\,.\,3}$, and so on; hence the series

$$1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \frac{x^5}{5\,!} + \dots$$

is a series which, when differentiated, gives as result

$$0 + 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \dots,$$

the very same series,—provided both extend to infinity: a very curious case, **the rate of variation of the series is equal to itself.** (Cf. p. 411.)

Such a series must therefore be appropriate for use in the theory of leaks, that is for dealing with a quantity whose rate of change is proportional to or equal to itself. We can guess therefore that such a series must, when plotted, give a curve of the nature of the exponential or logarithmic or geometrical-progression or compound-interest curve. If we call its value y, it satisfies the equation $\frac{dy}{dx} = y$ (cf. page 405).

It is a notable series. It is plainly convergent if x is less than 1; but it is convergent even when x is equal to 1 or greater than 1, because the denominators increase so fast; they increase so fast indeed that a moderate number of terms are generally sufficient to evaluate it fairly. The *powers* of x grow fast in size when x is greater than 1, but the *factorials* of the corresponding index-number grow still faster, and so must ultimately get bigger; for x stays as a constant factor while being raised to any power, while in 'factorials' the factor keeps on increasing. See page 315.

Let us try what the value of this series is when $x = 1$:

$$1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \tfrac{1}{720} + \tfrac{1}{5040} + \dots .$$

Greater than 2 and apparently less than 3, because

$$1 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \dots \text{ would equal } 3.$$

With patience its value can be reckoned to any desired degree of accuracy, and it comes out

$$2{\cdot}71828 \dots ,$$

a remarkable number, usually called e.

So now we can reckon what the series is when x has any other value than unity. If we try it arithmetically for $x = 2$ we shall get

$$1 + 2 + \tfrac{4}{2} + \tfrac{8}{6} + \tfrac{16}{24} + \tfrac{32}{120} + \tfrac{64}{720} + \dots ,$$

where we observe that though at first the numerators are bigger than the denominators, afterwards, in spite of the well-known rapid increase of the powers of 2, the factorials in the denominators soon overpower them; for $2^{12} = 4096$, whereas $\underline{|12} = 479{,}001{,}600$, and is thus a hundred thousand times as great.

To get a good value for this last series we must take a fair number of terms, ten or a dozen, into account; and if we do we find the result

$$7{\cdot}389\ldots,$$

which is e^2.

Similarly if we put $x = 3$ we shall get $20{\cdot}09\ldots$, which is e^3;
whereas if we put $x = \frac{1}{2}$ we get $1{\cdot}6467\ldots$, which is $\sqrt{e}$.

Thus we suspect that the series

$$1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots$$

is in fact e^x; which is true, and it is called the exponential series accordingly.

It has the singular and very useful property that its rate of change is equal to itself, that is to say that

$$\frac{d}{dx}e^x = e^x,$$

as we have already proved by differentiating each term of its series separately and observing that the series is unchanged by the process, being simply repeated over again.

Natural base of logarithms.

We can now apply this to logarithms:

Let $$y = e^x,$$

so that $$\log y = x \log e, \quad \text{or } \log_e y = x;$$

we have just learnt that in this case

$$\frac{dy}{dx} = y,$$

wherefore $$d \log_e y = dx = \frac{dy}{y}.$$

That is to say, the rate of change of the logarithm of a variable number is equal to the rate of change of the number itself divided by that number; provided the base of logarithms is e.

If we take any other base than e we shall not get quite so neat a result.

For let $u = r^x$, where r is any number whatever,

then $\log u = x \log r$ (or $\log_r u = x$),

and so $\frac{du}{u} = dx \,.\, \log r$;

wherefore $d \log_r u = dx = \frac{du}{u \log r}$,

which only reduces to the above simple form when $r = e$; otherwise it requires the natural logarithm of r to appear.

For instance, suppose we put $u = 10^{2 \cdot 9180303}$, as a representation of r^x, and change the index by a small amount, say to 2·9180408; then, by referring to an ordinary seven-figure table of logarithms, we shall see that the corresponding change in the number u is ·02

$$\text{since } u = 828 \cdot 00 \text{ and } u' = 828 \cdot 02.$$

Now our assertion is that the change in the logarithm ($x' - x$ or dx, viz. ·0000105) would have been equal to the change in the number ($u' - u$ or du) divided by the number (u or u'), that is to say would be practically equal to $\frac{\cdot 02}{828}$, if the base had been e; but since the base is 10 this result has to be divided by the further fixed quantity—the natural logarithm of our artificial base 10 (which is a number approximately equal to 2·3), in order to give the right result.

And it will be found accordingly that

$$\frac{\cdot 02}{828 \times 2 \cdot 3} = \cdot 0000105, \text{ almost exactly};$$

which illustrates the last algebraic line above.

Let us illustrate the occurrence of this natural logarithm of 10 by another numerical example, and at the same time make an estimate of its value.

Suppose we put 10^2 to represent r^x, and then allow the index x to increase somewhat, say to 2·01; what will be the corresponding change in r^x?

We might write $u = 10^{2\cdot00}$, $u' = 10^{2\cdot01}$;

so that $$\frac{du}{u} = \frac{u'-u}{u} = \frac{10^{2\cdot01} - 10^2}{10^2} = 10^{\cdot01} - 1. \quad \text{.........(1)}$$

But in general when $u = r^x$, $\dfrac{du}{dx} = r^x \log r$, wherefore

$$\frac{du}{u} = dx \, . \log r = \cdot01 \log 10; \quad \text{.........(2)}$$

and from these two expressions for the same thing we can approximately evaluate the natural log 10. For equating (1) and (2), we get

$$\log 10 = \frac{10^{\cdot01} - 1}{\cdot01} = 100 \times (10^{\cdot01} - 1)$$

$$= 100(\sqrt[100]{10} - 1) = 100(1\cdot0232 - 1) = 2\cdot32,$$

the last digit being affected by an error caused by the increase in x not being infinitesimal.

This 2·3... is approximately the logarithm of 10 to a certain base which has not been artificially specified, and which therefore must have entered automatically and "naturally" without convention or artifice. What is that base?

It is a number such that if raised to the 2·3... power it will equal 10. Call it n, then

$$2\cdot3 \ldots \log_{10} n = \log_{10} 10 = 1,$$

or $$\log_{10} n = \frac{1}{2\cdot3\ldots} = \cdot434\ldots,$$

wherefore $$n = 2\cdot717\ldots;$$

which plainly points to e, with a deficiency of one part in two thousand, or a twentieth of one per cent., due to approximations.

Clearly therefore there is something peculiar about e as the base of an exponential system: it is simpler than any other, and it occurs automatically or naturally, unless we force some other base in; for when one finds that

$$\frac{dr^x}{dx} = r^x \log r,$$

whereas

$$\frac{de^x}{dx} = e^x,$$

it becomes apparent that the base here automatically indicated is such as to make $\log e = 1$.

The fact is that $\log r$, wherever it naturally occurs, means $\log_e r$, and not a logarithm to any base at random. There appears therefore to be a **natural base** for logarithms; in this respect differing entirely from the base or radix of the scales of notation in ordinary counting. Ten, or twelve, or any number, might be used for that—it was a pure convention; but though, as soon as we have adopted ten as the numeration base, ten becomes specially convenient for practical calculations by aid of logarithms also, yet ten is not the natural base of logarithms; nor is it the simplest base for an exponent. That property specifically belongs to the incommensurable number called e.

The expansion of any exponential, such as r^x, is now easily managed in terms of e; for r may be expressed as e^k, whence $r^x = e^{kx}$; and we already know that

$$e^{kx} = 1 + kx + \frac{k^2x^2}{\underline{|2}} + \frac{k^3x^3}{\underline{|3}} + \dots.$$

But since $r = e^k$ it follows that $k = \log_e r$, hence the above expansion may also be written as

$$r^x = 1 + x\log r + \frac{(x \log r)^2}{\underline{|2}} + \dots,$$

where the logarithms are all to the base e.

For the special case when x is infinitesimal, say dt,

that gives us $$r^{dt} = 1 + dt \,.\, \log r,$$

wherefore $$r^{dt} - 1 = dt \,.\, \log r,$$

which justifies a step assumed above (page 410); where, however, it happened that the dt had a negative sign.

The whole theory of leaks or cooling is now quite easy, after this incursion into the elements of pure mathematics: for given that any quantity p (say pressure or temperature or potential) changes at a rate proportional to itself, we can write down instantly the following equivalent expressions (t meaning time):

$$\frac{dp}{dt} = -kp,$$

$$\frac{dp}{p} = -kdt,$$

$$d \log p = -kdt,$$

$$\log p' - \log p = k(t - t'),$$

$$\log p + kt = \text{constant} = \log p_0,$$

$$\log \frac{p}{p_0} = -kt,$$

$$p = p_0 e^{-kt}.$$

All these are different modes of expressing the same physical fact: the law of a cooling body, or a leaking reservoir, or any other of the many cases where rate of change of a quantity is proportional to the quantity itself; and the last gives explicitly the value of that quantity at any instant, in terms of the initial value, the logarithmic decrement, and the time.

And this must be regarded as typical of the way in which general facts in Physics are simplified, summarised, and compactly treated, by aid of more or less easy mathematics.

So ends the present book, but in a subject like this there can be no termination; every avenue leads out into infinity and must be left with its end open. In no science are there any real boundaries. In an advanced book a subjective boundary may be reached, viz. the boundary of our present knowledge; but in an elementary book like the present that is immensely far away, and the only terminus that can here be reached is a terminus of print and paper.

APPENDIX.

I. Note on the Pythagorean Numbers (Euc. I. 47). See Chapter XXXI.

By the Pythagorean numbers I mean simply those triplets of integers which serve to express the relative lengths of the sides of commensurate right-angled triangles: numbers therefore which satisfy the conditions of Euclid I. 47, that any two of them are greater than the third, and that the sum of the squares of two of them equals the square of the third.

The only numbers mentioned in the text, page 272 are:

3, 4, 5; 5, 12, 13; and 8, 15, 17;

but there are innumerable others.

The subject is of no practical importance, and is only mentioned here as an example of an easy kind of investigation in pure mathematics which an enthusiastic and advanced pupil might be encouraged to undertake, and which might lead him to take some interest in less simple parts of the theory of numbers. The result of the investigation, in this case, might be worded thus:

In general the sides and hypothenuse of a right-angled triangle are incommensurable, but an infinite number of such triangles exist in which the three sides may be represented by integers. These are of some interest, and the simplest of them, when the sides are in the ratio of the numbers 3, 4, 5, is frequently used by surveyors.

A formula from which all such sides may be calculated is the identity $$a^2b^2 + \left(\frac{a^2 - b^2}{2}\right)^2 \equiv \left(\frac{a^2 + b^2}{2}\right)^2;$$

meaning that ab and $\frac{1}{2}(a^2-b^2)$ represent the sides containing the right angle, and that $\frac{1}{2}(a^2+b^2)$ represents the hypothenuse.

To get a number of these triangles rapidly, without repetition of shape, *i.e.* without obtaining mere multiples of other sets, it is sufficient to choose as the auxiliary integers a and b any odd numbers which are prime to each other. The reason for choosing the auxiliary integers, a, b, as odd numbers prime to each other, is simply that if they contained a common factor the triplets obtained from them would be merely a multiple set representing the same shape as a simpler set; whereas if one was even while the other was odd, then a^2-b^2 would be odd, and $\frac{1}{2}(a^2-b^2)$ would not be an integer; or if everything were doubled it would be merely repeating the sides of some previous shape in another order.

Excluding multiple sets, one of the sides and the hypothenuse are always represented by an odd number, and the other side by an even number.

It is easy to prove that one of the sides containing the right angle must always be a multiple of 4, that one of them (it may be the same) must be a multiple of 3, and that one of the three sides (again it may be the same) must be a multiple of 5:

One of the two sides must be a multiple of 3.

Of course a, or b, may itself be a multiple of 3, thus satisfying the condition for the side ab. If neither of them is, then ab is not a multiple of 3, but in that case their squares must be of the form $3m+1$, $3n+1$ [or rather of the form $6m+1$, $6n+1$, since they are odd numbers], and so the other side, viz. $\frac{1}{2}(a^2-b^2)$ is then of the form $3(m-n)$.

One of the two sides must be a multiple of 4.

Taking
$$a = 2m+1,$$
$$b = 2n+1,$$
$$\tfrac{1}{2}(a^2-b^2) = 2(m-n)(m+n+1),$$
and either $m-n$ or $m+n+1$ must be an even number, since their difference is an odd number.

One of the three sides must be a multiple of 5.

If a or b is a multiple of 5, one of the sides, viz. the odd side, is the required multiple. If not, its square must be of the form $40m+9$ or $40m+1$. If the squares of both have the same remainder, the even side is a multiple of 5. If one has remainder 9 and the other 1, the hypothenuse is a multiple of 5, of the form $20m+5$.

So if neither a or b is a multiple of either 3 or 5 it follows that the number representing the even side has all three of the factors, 3, 4, 5 ; *i.e.* that it is a multiple of 60.

Moreover it can be shown that the hypothenuse is always itself the sum of two square numbers, one odd and one even, and that the odd side is the difference of those same squares. Thus, writing the odd side as

$$(2m+1)(2n+1) = (m+n+1)^2-(m-n)^2,$$

the even side is $$2(m-n)(m+n+1),$$

and the hypothenuse is $(m-n)^2+(m+n+1)^2$.

The following is a table of the Pythagorean triplets, with the mode of obtaining them displayed.

Auxiliary pair of numbers.	Odd side.	Even side.	Hypothenuse.
3, 1	3 = 4 − 1	4	5 = 4+1
5, 1	5 = 9 − 4	12	13 = 9+4
7, 1	7	24	25 = 16+9
9, 1	9	40	41 = 25+16
11, 1	11	60	61 = 36+25
13, 1	13	84	85 = 49+36
15, 1	15	112	113 = 64+49
17, 1	17	144	145 = 81+64
19, 1	19 = 100 − 81	180	181 = 100+81
5, 3	15	8	17 = 16+1
7, 3	21	20	29 = 25+4
9, 3	27	36	45 etc.
11, 3	33	56	65
13, 3	39	80	89
15, 3	45	108	117
17, 3	51	140	149
19, 3	57	176	185
7, 5	35	12	37
9, 5	45	28	53
11, 5	55	48	73
13, 5	65	72	97
15, 5	75	100	125
17, 5	85	132	157
19, 5	95	168	193

Auxiliary pair of numbers.	Odd side.		Even side.	Hypothenuse.	
9, 7	63		16	65	
11, 7	77		36	85	
13, 7	91		60	109	
15, 7	105		88	137	
17, 7	119		120	169	
19, 7	133		156	205	
11, 9	99		28	101	
13, 9	117		44	125	
15, 9	135		72	153	
17, 9	153	= 169 − 16	104	185	= 169 + 16
19, 9	171		140	221	
13, 11	143		24	145	
15, 11	165		52	173	
17, 11	187		84	205	
19, 11	209		120	241	

The left-hand column is simply a series of pairs of odd numbers, mainly prime to each other (but a few are included, for the sake of systematic completeness, which are not prime, and therefore involve repetition) ; the second column is their product ; the third column half the difference of their squares ; and the fourth column half the sum of their squares; the incipient columns merely illustrate the fact that the hypothenuse is the sum of two square numbers, one odd, one even, whose difference is equal to the odd side.

The identity $$(m^2-n^2)^2+(2mn)^2 \equiv (m^2+n^2)^2$$

represents the facts most simply, where m and n are any integers. One of these integers must be even and the other odd, with no common factor, if mere multiples or repetitions of shape are to be avoided.

II. Note on Implicit Dimensions (*see pp.* 53, 111, 143, 230).

The treatment of algebraical symbols as representing concrete quantities, with all the simplification and increased interest which this treatment involves, was first effectively called attention to by my brother, Alfred Lodge, at that time Professor of Pure Mathematics at Coopers Hill, and now a Mathematical Master at Charterhouse. See *Nature* for July, 1888, vol. 38, p. 281, which was the sequel to a pioneer paper read by him before the Association for the Improvement of Geometrical Teaching, in January, 1888.

The subject was subsequently and independently developed by Mr. W. Williams of South Kensington, now of Swansea; and, whether it has received full recognition or not, it has undoubtedly justified itself in the eyes of all who have put it to the test of practical experience. The whole subject is too large for this place, but a few elementary remarks are appropriate:

Quantities of different kinds do not occur in one expression; in other words, the terms of an expression must all refer to the same sort of things, if they are to be dealt with together or equated to any one thing. Nevertheless an expression like

$$x^3+5x^2+2x+6$$

is common, and x may be a length; which looks as if we could add together a volume, an area, a length, and a pure number. Not so, really, however; suppressed or implicit or unexpressed or masked dimensions must in that case exist in the numerical coefficients; the coefficient 5 must implicitly or tacitly refer to a length, 2 to an area, and 6 to a volume, if x is a length; and thus all the terms are really of the same kind. So they always will be in every real problem.

When an equation contains terms of essentially different kind, it must really consist of two or more equations packed together into the apparent form of one. Thus $\surd(-1)$ is a quantity of essentially different kind from 1 or $(\surd+1)$; the former being imaginary, the latter real. Hence if ever they occur together in an equation, as for instance in such an equation between complex quantities as

$$a\surd(+1)+b\surd(-1) = c\surd(+1)+d\surd(-1),$$

or what is the same thing (writing $\surd(-1)$ as i, for short, and $\surd(+1)$ as an unexpressed unity factor)

$$a+bi = c+di,$$

it must be regarded as a double equation, unless some of the quantities a, b, c, d are themselves complex; for it can only be interpreted and satisfied by the two separate equations

$$a = c \text{ and } b = d.$$

In other words it is really two equations packed together for brevity into a single statement.

For if either of these conditions is not satisfied, if for instance b is less than d, it is impossible to fill up the deficit by any increase in the value of a, since that refers to a quantity of totally different kind. A deficiency of oxygen in the atmosphere cannot be compensated by a surplus of gold in a bank; nor can deficiency of beauty be effectively counterbalanced by excess of size.

The group met with in a German philosophical treatise (according to a writer in the *Hibbert Journal*), as representing the class which does not "count" for moral and intellectual purposes,

"cows, women, sheep, Christians, dogs,
Englishmen, and other democrats,"

cannot be regarded as classified according to a satisfactory system, any more than can the somewhat similar group of tax-payers which is at present disfranchised by Act of Parliament.

So that any conclusions, inferences, or results due to the aggregation of such individuals in a community must be separable into a series of independent conclusions, inferences, or results, except in so far as some of these things are themselves complex, partaking more or less of each other's characteristics.

Sometimes we have equations among integers or other commensurable numbers, with incommensurables likewise involved, such as

$$m+n\sqrt{2} = x+y\sqrt{2}.$$

If now m, n, x, y are all to be considered integers or any vulgar fractions or terminating or recurring decimals, *i.e.* unless some of the quantities m, n, x, y are in whole or in part themselves surds, it must follow from the above statement that

$$x = m \text{ and } y = n,$$

otherwise the equation cannot be satisfied.

Again suppose x means a distance measured horizontally, and y a distance measured vertically, and the equation given is

$$ax+by = cx+dy;$$

it consists of two distinct and independent equations, unless a, b, c, d are themselves directional quantities and not mere numbers; in that case, however, *i.e.* in case a, c are vertical lengths and b, d are horizontal lengths, the equation is quite homogeneous and satisfactory, and denotes certain relations among rectangular areas. Or a, c may be reciprocals of horizontal lengths, and b, d reciprocals of vertical lengths; and so on. But if a, b, c, d are mere numbers, we are bound as before to equate the coefficients, that is to say to admit that $a = c$ and $b = d$; for no amount of horizontal travel is equivalent to a rise, nor can horizontal dimension make up for a deficiency in height.

In any single equation therefore, like $v^2 = 2gh$ for instance, where one side is plainly the square of a velocity, the other side must also be really, though not obviously, the square of a velocity. And since g is an acceleration and h is a height, those who know any mechanics will realise that the necessary condition is thoroughly satisfied.

But when g is interpreted as 32 or 981, the fact is masked, as facts often are masked by the incomplete method of arithmetical or numerical specification. If 32 or 981 is regarded as a pure number, which is all of g that it is customary actually to express in writing, then the equation becomes an absurdity, since it appears to assert that a velocity multiplied by itself results in a certain multiple of an elevation.

But when it is remembered that the 32 means really 32 feet per second per second, everything is perfectly right; for, the height being expressed in feet, the right-hand side of the above equation is so many square feet per second per second, or square feet divided by square seconds, which is the square of a velocity, in perfect agreement with the left-hand side.

So also in the equation to a parabola, $y = a + bx + cx^2$, the convention is that y is a vertical height, and x^2 the square of a horizontal length; but, since all the terms must really be alike in kind, it follows that a must be a vertical height (and it is: viz. the intercept on the vertical axis), that b must be a ratio of vertical to horizontal (and it is the tangent of an angle accordingly, namely the value of $\frac{dy}{dx}$ at the place where the curve cuts the axis of y): and further that c must be a sort of curvature, a quantity involving a vertical direction once in the numerator and a horizontal dimension twice in the denominator. It is in fact half $\frac{d^2y}{(dx)^2}$; it represents the rate at which the tangent to the curve swings round as the ordinate travels uniformly along the axis of x; and this rate, when measured by changes in the tangent of the angle of slope, is constant. Compare Chapter XXXIX.

But $y = x$ is also a possible equation, and looks as if a vertical height could be equivalent to a horizontal length. But it is only an appearance, due to suppressed quantities. The coefficient 1, not written, is really the tangent of an angle of 45°, and involves the ratio of vertical to horizontal required to restore the balance and common sense.

So also when $y = x^2$ there is an unwritten unity coefficient which is not a pure number, but an actual quantity, the ratio $y : x^2$, which the equation asserts has in this case the magnitude 1.

Or when $x = 6$, if x is a length, it follows that the 6 is a numerical abbreviation for 6 feet or 6 centimetres or 6 miles, measured in the same direction as x. See Article in "School World" for July, 1904.

It is frequently best to express these units fully, and not to get too exclusively into the habit of writing a length as 50 without saying whether inches or centimetres is intended, or an age as 15 without saying years or months, or a price as 42 without saying shillings or pounds, or whether it is per hundredweight or per ton (compare pages 232, 4). For though these and other less customary abbreviations are permissible among experts, beginners who get too used to them are apt to degenerate into slovenly incompleteness and inaccuracy, and to suffer by finding difficulties hereafter where none exist.

III. Note on Factorisation (see Chapter XIV.).

A quadratic expression ax^2+bx+c can be resolved into factors if the middle term bx can be separated into two parts such that when multiplied together the product is acx^2.

Thus take $3x^2+10x+7$,
and write it $3x^2+7x+3x+7$;
it becomes at once $(3x+7)(x+1)$.

Again take $5x^2+27x+28$,
and write it $5x^2+7x+20x+28$;
it becomes $(5x+7)(x+4)$.

When a quadratic expression is thus written in four terms, such that the product of the means is equal to the product of the extremes, the four terms are necessarily proportional; and if such proportionality does not hold, you cannot factorise.

When they are proportional, as in the above case, and their sum equated to zero, $(5x^2+7x)$ and $(20x+28)$ must have a common factor ; so also must $(5x^2+20x)$ and $(7x+28)$ have another common factor.

If we write such four proportional terms with the common factors displayed, they must have the form

$$ac+ad+bc+bd\,;$$

which terms geometrically represent themselves thus :

a	b	
ac	bc	c
ad	bd	d

In the above example either a is x and b is 4, which are equivalent ; or a is $5x$ and b is 7, which are also equivalent. For instance the diagram explicitly applied to the above case would look thus : no scale being implied in the drawing, but simply a framework.

$(5x)$	(7)	
$5x^2$	$7x$	(x)
$20x$	28	(4)

IV. Note on the Growth of Population (page 220).

In spite of what has been said in the text as to the danger of applying the geometrical law of increase, or indeed any fixed law of increase, to a given country or to any assemblage without taking into account all the circumstances, nevertheless the growth of the population of England and Wales during the last century illustrates with remarkable closeness the geometrical-progression law.

The following is a table of the common logarithms (to base 10) of the population of England and Wales for each decade from 1801 to 1901 ; together with their differences. If the geometrical law held precisely, these differences would all be constant ; as it is, they hover about a mean value, except in some of the early years of the century, when they are abnormally big,—apparently as a reaction from the Napoleonic wars, but doubtless also on account of some applications of science, and other economic conditions.

Population table of England and Wales for last century.

	Logs.	Diffs.
1801	6·949026	
		·058050
1811	7·007076	
		·072114
1821	7·079190	
		·063724
1831	7·142914	
		·058869
1841	7·201783	
		·051739
1851	7·253522	
		·048944
1861	7·302466	
		·053794
1871	7·356260	
		·058286
1881	7·414546	
		·047890
1891	7·462436	
		·049796
1901	7·512232	

From these data the curve of population might be plotted, and it will be seen that from 1841 onwards it would be fairly steady, the mean or average difference for 10 years during this period being ·051741. So the difference of logarithms for one year is ·00517, or to base e, ·0119, or say ·012. But we know that $d \log_e p = \frac{dp}{p}$ (see Chap. XLII.); hence ·012, or 12 per thousand per annum, is the average rate of increase of the population since 1841. The curve is mainly a geometrical-progression or exponential curve, with this value as the common ratio.

All the fluctuations noticed in such a curve could doubtless be explained instructively, though to some extent hypothetically, by a Historian. For instance there is an excessive rate of growth in the decade 1871 to 1881, which probably includes a period of good trade; but even that is not equal to the rates of increase nearer the beginning of the century, when presumably the population was emerging out of extreme poverty.

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TABLE OF 3-FIGURE LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9
1·0	000	004	009	013	017	021	025	029	033	037
1·1	041	045	049	053	057	061	065	068	072	075
1·2	079	083	086	090	093	097	100	104	107	111
1·3	114	117	121	124	127	130	134	137	140	143
1·4	146	149	152	155	158	161	164	167	170	173
1·5	176	179	182	185	188	190	193	196	199	201
1·6	204	207	210	212	215	218	220	223	225	228
1·7	230	233	236	238	241	243	246	248	250	253
1·8	255	258	260	263	265	267	270	272	274	277
1·9	279	281	283	286	288	290	292	295	297	299
2	301	322	342	362	380	398	415	431	447	462
3	477	491	505	519	532	544	556	568	580	591
4	602	613	623	634	644	653	663	672	681	690
5	699	708	716	724	732	740	748	756	763	771
6	778	785	792	799	806	813	820	826	833	839
7	845	851	857	863	869	875	881	887	892	898
8	903	909	914	919	924	929	935	940	945	949
9	954	959	964	969	973	978	982	987	991	996

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